INTEGRATED GANIT

Class IX X
VOL-II

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INTEGRATED GANIT
(CLASSES IX & X)
VOLUME II

GEOMETRY, MENSURATION
& TRIGONOMETRY

This is the only Text Book on Mathematics
for Classes IX & X of all affiliated School

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FOREWORD

The existing Syllabi of Mathematics for Classes VI to X in the Schools recognised by the Board have been running in phases since 1981. Accordingly, the text books of Mathematics for Classes VII to X are published by the Board. The text book of Mathematics for Class VI is the publication of Govt. of West Bengal. The Government, however, arranges for free distribution of the book.

Necessity had been duly felt to rearrange the contents of the erstwhile two different Mathematics books, one each for Class IX and Class X. This was particularly in consideration that the Syllabus of Mathematics as the Syllabi of all other Subjects for Madhyamik Priksha of the Board is an integrated one for Classes IX and X. Consequently, as well as in view of the volume of the book, the Mathematics text book for Classes IX and X taken together is now published in two parts — Part I consisting of Algebra and Arithmetic, Part II of Geometry, Trigonometry and Mensuration.

This has been a matter of privilege to the Board that many teachers, educationists and also some students had suggested some corrections in our text books on Mathematics. Efforts have been made to incorporate the corrections as well as to minimise certain shortcomings to uplift the standard of the books.

In this context I like to impress upon of all concerned that the books prepared by the Board are the only text books on Mathematics approved by the Board. The School authorities are, therefore, requested not to put the guardians and their wards in confusion by prescribing any other Mathematics book whatsoever as text books for Classes VI to X.

I look forward for the cooperation of teachers, educationists and students also in future as in the past.

Chittaranjan Bandopadhyay
President,
West Bengal Board of
Secondary Education.

77/2, Park Street,
Calcutta - 700 016,
The 17th July, 1992.

Revised Syllabus in Mathematics for classes IX & X
[Geometry, Mensuration & Trigonometry]

Geometry **Marks 30**

1. Revision of previous works through simple exercises.

2. To establish the following propositions :
   - (a) Each diagonal of a parallelogram divides the parallelogram into congruent triangles.
     The popposite sides and angles of a parallelogram are equal and the diagonals bisect one another.
   - (b) A quardrilateral is a parallelogram if
     - (i) opposite sides are equal, or
     - (ii) opposite angles are equal, or
     - (iii) any two opposite sides are equal and parallel, or
     - (iv) its diagonals bisect one another.
   - (c) If there are three or more parallel lines and the intercepts made by them on any one straight line that cuts them are equal, then the corresponding intercepts on any other straight line that cuts them are also equal.
   - (d) The straight line, drawn through the middle point of one side of a triangle parallel to another side, bisects the third side and is equal to half of the second side. The line segment joining the middle points of two sides of a triangle is parallel to the third side and is equal to half of it.
   - (e)
     - (i) Parallelograms on the same base and between the same parallels (or of the same altitude) are equal in area.
     - (ii) Triangles on the same base (or on equal bases) and between the same parallels (or of the same altitude) are equal in area.
     - (iii) Triangles having equal area, on the same base and on the same side of it, are between the same parallels.
     - (iii) If a triangle and a parallelogram stand on the same base and between the same parallels, the area of the triangle is half that of the parallelogram.
   - (f)
     - (i) The perpendicular bisectors of the sides of a triangle are concurrent.
     - (ii) The perpendiculars drawn from the vertices of a triangle on the opposite sides are concurrent.
     - (iii) The bisectors of the angles of a triangle are concurrent.
     - (iv) The medians of a triangle are concurrent.

3. 'Pythagoras' Theorem—Statement and demonstration through activity only.

4. Constructions with proofs :
   - (i) To draw a parallelogram equal in area to a given triangle and having one of its angles equal to a given angle.
   - (ii) To draw a triangle equal in area to a given quardrilateral.

5. Simple problems based on the above propositions and constructions.

6. Revision of previous work through simple exercises.

7. To prove that
   - (a) There is one circle and only one which passes through three given points not in a straight line.
   - (b) A straight line drawn from the centre of a circle to bisect a chord which is not a diameter is at right angles to the chord, and conversely.
   - (c) The angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference.
   - (d) Angles on the same segment of a circle are equal. If a line segment joining two points subtends equal angles at two other points on the same side of it, the four points shall lie on a circle.
   - (e) The angle in a semi-cirlce is a right angle.
   - (f) The opposite angles of any quadrilateral inscribed in a circle are supplementary, and conversely.
   - (g) (i) The tangent at any point of a circle and its radius through that point are perpendicular to one another.
     - (ii) The segments of two tangents of a circle from an external point to the points of contact are equal and they subtend equal angles at the centre.
     - (iii) If two circles touch, the point of contact lies on the straight line through the centres.

8. Properties of Ratio and Proportion with reference to geometrical figures.

9. To prove that :
   - (a) If a straight line is drawn parallel to one side of a triangle the other two sides are divided proportionally, and the converse. (Proof based on the area proposition should be used.)
   - (b) If two triangles are equiangular, their corresponding sides are proportional, and conversely.
   - (c) If a perpendicular is drawn from the vertex containing the right angle of a right-angled triangle, on the hypotenuse, the triangles on each side of the perpendicular are similar to the whole triangle and to one another.
   - (d) Pythagoras' theorem and its converse.

10. Constructions with proofs :
    - (i) To draw a circle circumscribing a triangle.
    - (ii) To draw a circle inscribed in a triangle.
    - (iii) To draw a tangent to a circle :
        - (a) from an external point,
        - (b) at a point on the circle.
    - (iv) To draw common tangents to two given circles.
    - (v) To draw the mean proportional of two line segments.
11. Simple problems based on the above propositions and constructions.

Mensuration **Marks 15**

1. Perimeter and area of a rectangle, a square, a triangle. Area of any rectilinear figure.
2. Circumference and area of a cirlce. (The approximate value of π to be taken as $\frac{22}{7}$)

   (only statement of formulae and their numerical applications)
3. Revision of previous work through simple exercises.
4. Surface and volume of
    - (i) A rectangular parallelopiped,
    - (ii) a right circular cylinder, and a right circular cone, and
    - (iii) a sphere.

Trigonometry **Marks 15**

1. Idea of trigonometrical angles :
   Positive and Negative angles. Measurement of angles in Sexagesimal and Circular measures only. (π radian = 180° to be assumed).
2. Definition of Trigonometrical ratio of an acute angle, Trigonometrical ratios of the following angles : 0°, 30°, 45°, 60°, 90°, (except undefined values such as tan 90°, cot 0°).
3. Trigonometrical ratios of complementary angles.
4. Simple problems on heights and distances reducible to the solution of right-angled triangles involving the angles mentioned above.

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GEOMETRY

CHAPTER 1

Revision Exercises on Previous Lessons

Exercise 1

(In every case give reasons in support of your answer)

1. ABC is a right-angled triangle in which $\angle A$ is a right angle and $\angle C = 55°$; DE is drawn perpendicular to the side BC from any point D on the side AB. What is the measure of $\angle BDE$? What is the relation between $\triangle ABC$ and $\triangle DBE$?

2. (i) What is the sum of the interior angles of a polygon of 10 sides? If the sides of this polygon be produced in the same order and in the same direction, what will be the sum of the exterior angles so formed? Had the number of sides of the ploygon been 12, what would have been the sum of the interior and that of the exteiror angles?

(ii) What is the number of sides of a regular ploygon each of whose exterior angles is 40°?

(iii) What is the number of sides of a regular polygon each of whose interior angles is 150°?

(iv) What are the measures of each interior and each exterior angle of a regular polygon of 9 sides?

3. Side BC of $\triangle ABC$ is produced to the point D. $\angle ACD = 112°$, and $\angle B = \frac{3}{4} \angle A$; find the value of $\angle A$.

4. In the isosceles triangle ABC, $AB = AC$, $\angle A = 70°$; bisectors of the two exterior angles obtained by producing BC on both sides meet at the point O. What is the measure of $\angle BOC$? What type of quadrilateral is $ABOC$?

5. In which of the following cases will the three points P, Q, R be collinear?

(i) $QP = QA = QR$; $\angle AQP$ and $\angle AQR$ are two adjacent angles; $\angle APQ = 70°$, $\angle QAR = 45°$.

(ii) $QP = QA = QR$; $\angle AQP$ and $\angle AQR$ are two adjacent angles; $\angle APQ = 70°$, $\angle QAR = 20°$.

(iii) An equilateral triangle APQ and an isosceles triangle AQR, whose side AQ = side QR, are such that $\angle AQP$ and $\angle AQR$ are adjacent angles; $\angle QAR = 30°$.

(iv) An equilateral triangle APQ and an isosceles triangle AQR, whose side AQ=side QR, are such that $\angle AQP$ and $\angle AQR$ are adjacent angles; $\angle QAR=40°$.

(v) Q is the mid-point of the line segment AB; two line segments AP and BR are drawn in such a way that they are on opposite sides of the line segment AB and $\angle PAQ=\angle RBQ$.

(vi) Q is the mid-point of the line segment AB; two equal and parallel line segments AP and BR are so drawn that they are on opposite sides of the line segment AB.

(vii) In $\triangle APQ$, $\angle APQ=60°$ and $\angle PAQ=30°$; in $\triangle AQR$, $AQ=QR$ and $\angle QAR=45°$; $\angle AQP$ and $\angle AQR$ are two adjacent angles.

(viii) QST is a triangle. $\angle PQT = \angle QTS$ and they are on opposite sides of QT. $\angle RQS=\angle QST$ and they are on opposite sides of QS.

6. Side BC of the equilateral triangle ABC is produced to the point D such that $CD=AB$. What is the measure of $\angle BAD$?

7. In $\triangle ABC$, $AC=BC$ and $\angle ABC=70°$. BC is produced to the point D such that $BC=CD$. What is the measure of $\angle BAD$?

8. In $\triangle ABC$, $AB=AC$ and $\angle ACB=50°$. CA is produced to the point D such that $\angle ABD=40°$. What is the relation between AB and AD?

9. In the parallelogram $ABCD$, $\angle BAD=45°$; find the measure of each of the other three angles.

10. $ABCDEF$ is a hexagon and O is an interior point. If the angles subtended by the sides of the hexagon at the point O are equal to each other, find the measure of each angle. Which line segments through the point O will lie on the same straight line? If those line segments are of equal length, what type of polygon will $ABCDEF$ be?

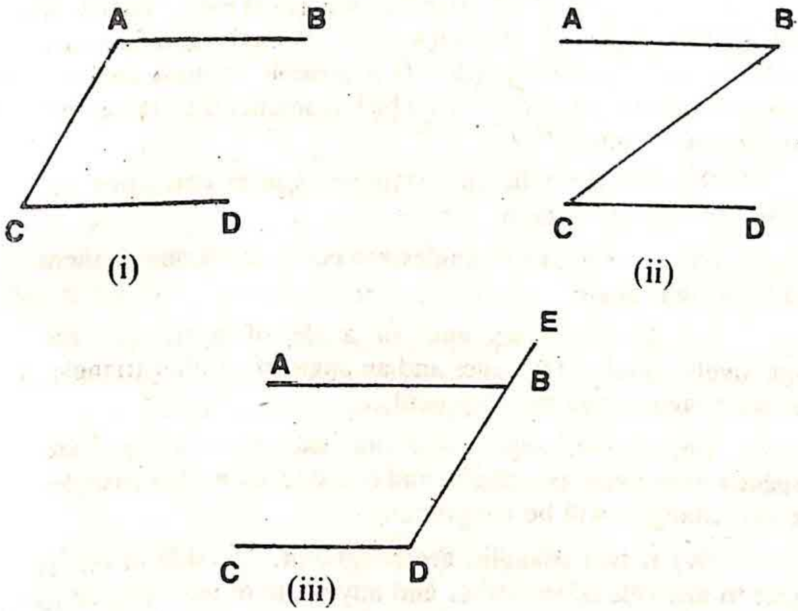
11. (i) In $\triangle ABC$, $\angle ABC=50°$, $\angle BCA=60°$. Arrange the sides of the triangle in descending order of their lengths.

(ii) In $\triangle ABC$, $AB=5$ cm, $BC=4$ cm and $CA=7$ cm. Arrange the angles of the triangle in ascending order of their magnitudes.

12. Two line segments AB and CD intersect each other the point O.

If $\angle AQD = \frac{3}{2} \angle DOB$, what is the measure of each angle?

13. In the following figures (without producing the line segments to any direction) find the condition or conditions under which the two line segments AB and CD will be parallel to each other.



(i) (ii) (iii)

Fig. 1

14. In which of the following cases will it be possible to construct a quadrilateral ? In each of those possible cases the sides are to be taken in a certain order: state that order.

(i) Length of a diagonal is 7 cm and the lengths of the sides are 5 cm, 2 cm, 6 cm and 4 cm.

(ii) Length of a diagonal is 7 cm and the lengths of the sides are 3 cm, 5 cm, 4 cm and 2 cm.

(iii) Length of a diagonal is 7 cm and the lengths of the sides are 6 cm, 5 cm, 4 cm and 4 cm.

15. AB, BC and CA are three line segments. BC is greater than AB by 4 cm and CA is smaller than AB by 8 cm. The length of the line segment AB should be greater than at least how many

centimeters such that the three line segments can form a triangle?

16. AB is a straight line and O is an external point. P, Q, R, S are points on the straight line in the above order such that $\angle OPB=60°$, $\angle POQ=20°$, $\angle POR=30°$ and $\angle POS=40°$. Which of the four line segments OP, OQ, OR, QS is the smallest and which two are equal in length ? Is it possible to draw any line segment from the point O to AB which is smaller than the above smallest line segment ?

17. Rewrite the following statements after correcting the geometrical error in them.

(i) If two adjacent angles are equal, each one of them will be a right angle.

(ii) If two sides and an angle of a triangle are respectively equal to two sides and an angle of another triangle, the two triangles will be congruent.

(iii) If two angles and one side of a triangle are respectively equal to two angles and one side of another triangle, the two triangles will be congruent.

(iv) If two triangles are congruent, any side of one is equal to any side of the other and any angle of one is equal to any angle of the other.

(v) The straight line EF intersects two straight lines AB and CD at the points G and H respectively. (a) Since $\angle BGE$ and $\angle DHG$ are corresponding angles, AB and CD are parallel. (b) Since $\angle BGH$ and $\angle GHC$ are alternate angles, AB and CD are parallel.

18. (i) $\triangle ABC$ and $\triangle PQR$ are congruent as $AB=QR$, $BC=RP$ and $\angle ABC = \angle QRP$. Which of the other two angles of the first triangle are respectively equal to the other two angles of the second triangle?

(ii) $\triangle PQR$ and $\triangle XYZ$ are congruent as $\angle PQR = \angle YZX$, $\angle PRQ = \angle YXZ$ and $QR=ZX$. Which pairs of the other two sides of the two triangles are equal ?

(iii) $\triangle ABC$ and $\triangle XYZ$ are congruent as $AB=ZY$, $BC=YX$ and $CA=XZ$. Which pairs of the three angles of the two triangles are equal ?

Exercise 2

1. Draw a line segment AB of length 6 cm. At the point *A* draw one angle 60° with AB as one arm and cut off AC of length 4·5 cm from the other arm of the angle. Draw straight lines through the points *C* and *B* parallel to *AB* and *AC* respectively. These two lines meet at the point *D*. What type of quadrilateral is *ABDC* ?

2. Draw a line segment AB of length 8 cm. Take a point *C* outside *AB*. Draw a line through the point *C* parallel to *AB*. Take a point *D* on this parallel line. Join *A*, *C* and *A*, *D*. Through the point *B* draw two lines parallel to *AC* and *AD* intersecting the line *CD* at the points *E* and *F* respectively. Name the two parallelograms thus formed.

3. Draw any line-segment *AB*. Mark *O*, the midpoint of *AB* (by ruler and compasses only). Draw a line-segment *OC* through the point *O* and from *CO* produced cut off *OD* equal to *OC*. With the help of set-square can you say what is the relation between *AC* and *DB* ? What is the relation between *AD* and *CB* also ?

4. Construct a parallelogram *ABCD*, such that side, $AB=8{\cdot}5$ cm, side $AD=5{\cdot}7$ cm and $\angle BAD=60°$. Take a point *E* on *DC* and join *B*, *E*. Through the point *A* draw a line parallel to *BE* to intersect *CD* produced at the point *F*. Can you say what type of quardrilateral is *ABEF* ?

5. Construct a rhombus *ABCD* such that length of each side is 6 cm and $\angle BAD=45°$. Draw perpendiculars to *AB* at the points *A* and *B* to meet *CD* produced and *CD* at the points *F* and *E* respectively. Find, with the help of set-squares, the relation between the sides *AB* and *FE* of the quardrilateral *ABEF*. What type of quadrilateral is *ABEF* ?

6. Construct a triangle *ABC* such that $AB=7{\cdot}5$ cm, $BC=12{\cdot}5$ cm and $CA=10$ cm. Now construct a right-angled triangle *BCD* such that $\angle C$ is a right angle, $CD=6$ cm and the point *D* is on the same side of *BC* as the point *A*. Draw the line *AD*. With the help of set-squares find the relation between *AD* and *BC*

7. Construct a triangle *ABC* such that $AB=8$ cm, $BC=5$ cm and $\angle ABC=60°$. Draw perpendiculars from each vertex of the triangle to the opposite side. Observe how the three perpendiculars are situated.

8. Construct a triangle ABC such that $AB=3{\cdot}5$ cm, $BC=4$ cm and $CA=4{\cdot}7$ cm. Draw a straight line through the point A parallel to BC. Now through the point B draw a straight line parallel to CA to intersect the previous parallel straight line through the point A at the point P. Again through the point C draw a straight line parallel to AB to intersect the parallel lines through the points A and B at the points Q and R respectively. What types of quardrilaterals are $ABCQ$, $APBC$ and $ABRC$? Find with the help of compasses, the relations between the following pairs of the line segments:

(i) AP and AQ, (ii) BP and BR, (iii) CR and CQ, (iv) AB and QR, (v) BC and PQ, (vi) CA and RP.

9. Construct a triangle ABC such that $\angle ABC=75°$, $\angle BCA=45°$ and $BC=7$ cm. Draw the perpendicular bisectors of the sides of the triangle. Observe how the perpendicular bisectors are situated.

10. Construct a triangle ABC such that $\angle A$ is a right angle, hypotenuse $BC=10$ cm and $AB=6$ cm. Draw the internal bisectors of the angles of the triangle. Observe how these internal bisectors are situated.

11. Construct a triangle ABC such that $\angle BAC=105°$, $\angle BCA=45°$ and $AC=4{\cdot}6$ cm. Find the mid-points of the sides (with the help of ruler and compasses). Now draw the medians of the triangle. Observe how these medians are situated.

12. Construct a triangle ABC such that $\angle ABC=30°$, $AB=AC=9$ cm. Bisect BC. Through the point A draw a straight line parallel to BC. At the mid-point O of BC construct $\angle COD=60°$ such that OD intersects the line parallel to BC at the point D. Now through the point C draw a straight line parallel to OD to intersect the line parallel to BC at the point E. Can you say what type of quadrilateral is $CODE$?

13. Construct a quadrilateral $ABCD$, such that $AB=4$ cm, $BC=5$ cm, $CD=3$ cm, $DA=6$ cm and diagonal $AC=7$ cm. Draw a straight line through the point B parallel to AC.

14. Construct a square with side $6{\cdot}8$ cm.

15. In $\triangle ABC$, $\angle B$ is a right angle, hypotenuse $CA=7{\cdot}5$ cm and $BC=6$ cm. Construct three squares $ACKL$, $BCDE$ and $ABFG$ on the three sides of the triangle outside it.

Exercise 3

1. $ABCD$ is a parallelogram. Write its properties from the definition. By drawing the diagonal AC, prove that $\triangle ABC \cong ADC$.

2. Two line segments AB and DC are equal and parallel. AC and BD intersect each other at the point O. Prove that $AQ=OC$, $BO=OD$.

3. In the quadrilateral $ABCD$, $AB=DC$ and $AD=BC$; prove that $\triangle ABC \cong \triangle ACD$.

4. What is the sum of the four interior angles of a quadrilateral $ABCD$? If each pair of opposite angles of this quadrilateral are equal what will be the sum of $\angle A$ and $\angle D$? Can you now say what is the relation between AB and DC ? Give reasons.

5. In the quadrilateral $ABCD$, $AB=DC$ and $AB \parallel DC$. Prove that $\triangle ABC \cong \triangle ADC$.

6. Two line segments AC and BD bisect each other at the point O. Prove that $\triangle ABC \cong \triangle DOC$; hence prove that $AB=DC$ and $AB \parallel DC$.

7. In the quadrilateral $ABCD$, AB and DC are equal and parallel. Prove that $\triangle ABC \cong \triangle CDA$. Hence prove that AD and BC are also equal and parallel.

8. D is the mid-point of the side AB of $\triangle ABC$. A straight line through the point D intersects AC at the point E. From that straight line cut off DF making equal to ED. Join B, F. Prove that $\triangle ADE \cong \triangle BDF$. Hence prove that $EC \parallel FB$. Had the straight line drawn through the point D been parallel to BC, what type of quadrilateral would $BCEF$ have been ? Give reasons.

9. D and E are the mid-points of the sides AB and AC respectively of the trianlge ABC. Join D, E and produce it to the point F such that $EF=DE$. Join F, C. Prove that $\triangle ADE \cong \triangle CEF$ and hence prove that $BA \cong CF$.

10. ABC is a triangle. The perpendicular to AB at its mid-point D intersects the perpendicular to AC at its mid-point E at the point O. Join O, A; O, B and O, C. Prove that $\triangle AOD \cong \triangle BOD$ and $\triangle AOE \cong \triangle COE$. Hence prove that $OA=OB=OC$.

11. Internal bisectors of $\angle B$ and $\angle C$ of $\triangle ABC$ intersect each other at the point I. ID, IE and IF are perpendiculars drawn from the point I to the sides BC, CA and AB respectively. Prove that $\triangle CID \cong \triangle CIE$ and $\triangle BID \cong \triangle BIF$; hence prove that $ID=IE=IF$.

12. O is a point in the interior region of $\angle BAC$ such that OD and OE, perpendiculars drawn from the point O to AB and AC respectively, are equal. Prove that $\triangle OAD \cong \triangle OAE$.

13. AB and CD are two line segments. E and F are two points on the line segment CD in the order C, E, F, D, AE and BD are equal and parallel; AC and BF are also equal and parallel. Prove that $\triangle ACE \cong \triangle BFD$.

14. AP and BQ are perpendiculars to the line segment AB at the points A and B respectively. If $AP=BQ$, prove that $\triangle ABP \cong \triangle ABQ$; hence prove that $BP=AQ$.

15. AP and BQ are perpendiculars to the line segment AB at the points A and B respectively. AQ and BP intersect each other at the point R. If $AP>BQ$, prove that $\angle ABP > \angle BAQ$; hence prove that $AR>BR$.

16. B and C are two points on the line segment DE such that the four points are in the order D—B—C—E. $\triangle ABC$ is such that $AB=BD$ and $AC=CE$. Prove that $\angle ADB = \frac{1}{2} \angle ABC$, $\angle AEC = \frac{1}{2} \angle ACB$ and $\angle DAE=90^\circ + \frac{1}{2} \angle BAC$.

17. In $\triangle ABC$, $\angle A$ is obtuse. Side BA of $\triangle ABC$ is produced to the point D such that $AB=CD$. Prove that $\angle BAC=90^\circ + \frac{1}{2} \angle ADC$.

18. Side BA of $\triangle ABC$ is produced to the point D such that $AD=AC$. Prove that $\angle BAC = 2\angle ADC$.

CHAPTER 2

Theorems

2.1 Theorems on the properties of parallelogram :

The properties that we get from the definition of a parallelogram are that its opposite sides are parallel to each other. Apart from this there are other properties of a parallelogram also, which we will get from the following theorems.

Theorem 17

Any diagonal of a parallelogram divides it into two congruent triangles and opposite sides and angles of a parallelogram are equal.

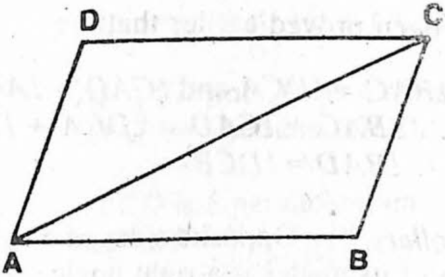


Fig. 2

Let $ABCD$ be a parallelogram, where $AB \parallel DC$ and $AD \parallel BC$.

Diagonal AC is drawn.

It is required to prove that

(i) $\triangle ABC \cong \triangle CDA$.

(ii) $AB=DC$ and $AD=BC$

(iii) $\angle BAD = \angle DCB$ and $\angle ABC = \angle CDA$.

Proof : $\because$ $AB \parallel DC$ and AC is a transversal

$\therefore$ $\angle BAC$=alternate $\angle DCA$.

Again, $\because AD \parallel BC$ AC is a transversal
$\therefore \angle ACB =$ alternate $\angle CAD$.

Now in $\triangle ABC$ and $\triangle CDA$
$\angle BAC = \angle ACD$ (proved)
$\angle ACB = \angle CAD$ (proved)
and AC is common

$\therefore \triangle ABC \cong \triangle CDA$ (i)

Again, $\because \triangle ABC \cong \triangle CDA$
$\therefore AB = DC$
and $AD = BC$ } (corresponding sides) (ii)

Again, $\because \triangle ABC \cong \triangle CDA$ (corresponding angles)

It has been proved earlier that

$$\angle BAC = \angle DCA \text{ and } \angle CAD = \angle ACB$$
$$\therefore \angle BAC + \angle CAD = \angle DCA + \angle ACD$$
$$\therefore \angle BAD = \angle DCB \qquad \text{(iii)}$$

Corollary : 1. Opposite sides of a rectangle are equal and each of its angles is a right angle

2. Sides of a square are all equal and each of its angles is a right angle.

Theorem 18

The diagonals of a parallelogram bisect each other.

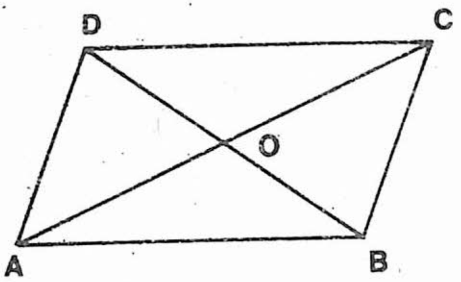


Fig. 3

Let $ABCD$ be a parallelogram where $AB \parallel DC$ and $AD \parallel BC$. The diagonals AC and BD intersect each other at the point O.

It is required to prove that $AO = OC$ and $BO = OD$.

Proof : $\because$ $AB \parallel CD$ and AC is a transversal.
$\therefore \angle OAB =$ alternate $\angle OCD$.

Again, $\because$ $ABCD$ is a parallelogram
$\therefore AB = DC$

Now in $\triangle AOB$ and $\triangle COD$
$\angle OAB = \angle OCD$ (proved)
$\angle AOB =$ vertically opposite $\angle COD$

and the two corresponding sides $AB = DC$

$\therefore \triangle AOB \cong \triangle OCD$
$\therefore AO = OC$ and $BO = OD$ (corresponding sides)

Some applications :

1. The bisectors of a pair of opposite angles of a parallelogram are parallel to each other.

Let the bisectors of $\angle A$ and $\angle C$ intersect DC and AB at the points P and Q respectively.

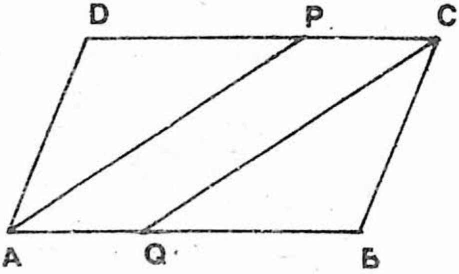


Fig. 4

It is required to prove that $AP \parallel QC$.

Proof : $\angle DPA = \angle PAQ$ (alternate) [$\because DC \parallel AB$]

$= \frac{1}{2} \angle DAB$

$= \frac{1}{2} \angle DCB$ [$\because$ opposite angles of a parallelogram and equal]

$= \angle DCQ$ (corresponding)

$\therefore$ *the straight line* DPC is a transversal of AP and QC and the two corresponding angles are equal.

$\therefore AP \parallel QC$

2. $ABCD$ is a parallelogram with $AB > AD$. Prove that $\angle BAC < \angle DAC$.

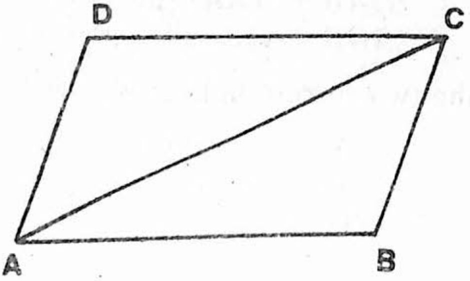


Fig. 5

Proof : $\because$ $ABCD$ is a parallelogram

$\therefore AD = BC$

Again, $AD \parallel BC$ and AC is a transversal

$\therefore \angle DAC$=alternate $\angle ACB$

$\because AB > AD$

$\therefore AB > BC$

$\therefore$ in $\triangle ABC$, $\angle BAC < \angle ACB$

$\therefore \angle BAC < \angle DAC$

3. ABCD is a parallelogram with $AB=2AD$. Prove that the bisectors of $\angle A$ and $\angle B$ meet at right angles at the mid-point of DC.

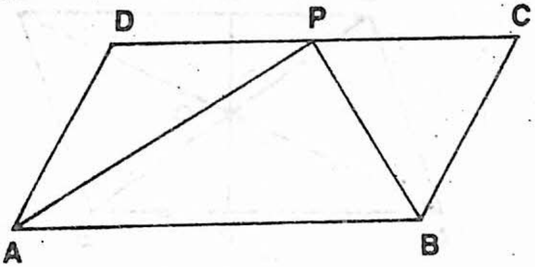


Fig. 6

Let the bisector of $\angle A$ of the parallelogram $ABCD$ meet DC at the point P. P, B are joined.

It is required to prove that (i) P is mid-point of DC (ii) BP is the bisector of $\angle B$ and (iii) $\angle APB$ =1 right angle.

Proof : $\because AP$ is the bisector of $\angle A$

$\therefore \angle DAP = \angle PAB$

$= \angle DPA$ (alternate) [$\because DC \parallel AB$]

$\therefore$ in $\triangle DAP$, $DP = DA = \frac{1}{2} AB = \frac{1}{2} DC$

$\therefore PC = DP = DA = BC$.

$\therefore P$ is mid-point of DC (i)

In $\triangle CPB$, $\angle CBP = \angle CPB$ [$\because PC = BC$]

= Alternate $\angle PBA$ [$\because DC \parallel AB$]

$\therefore PB$ is the bisector of $\angle ABC$ (ii)

Again, $\angle PAB + \angle ABP = \frac{1}{2}(\angle DAB + \angle ABC)$

$= \frac{1}{2} \times 180° = 90°$

$\therefore \angle APB = 180° - (\angle PAB + \angle ABP)$

$= 180° - 90° = 90° = 1$ right angle (iii)

4. Diagonals AC and BD of the parallelogram $ABCD$ intersect each other at the point O. A straight line through the point O intersects AB and DC at the points P and Q respectively. Prove that $OP=OQ$.

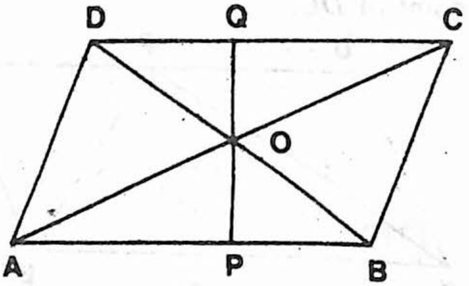


Fig. 7

Proof : In $\triangle PBO$ and $\triangle QDO$

$\angle PBO = \angle QDO$ (alternate) [$\because DC \parallel AB$]

$\angle POB = \angle QOD$ (vertically opposite)

and $OB=OD$ ($\because$ the diagonals of a parallelogram bisect each other)

$\therefore \triangle PBO \cong \triangle QDO$

$\therefore OP=OQ$ (corresponding sides)

Exercise 4

1. Without using diagonal prove, only from the properties of parallel lines, that opposite angles of a parallelogram are equal.

2. Prove that the quardrilateral formed by the bisectors of the four angles of a parallelogram is a rectangle.

3. Prove that the two diagonals of a rectangle are equal.

4. Prove that the diagonals of a square are equal and they bisect each other at right angles.

5. ABC is an equilateral triangle and D, E, F are the mid-points of BC, CA, AB respectively. Prove that $\triangle AEF$ and $\triangle BDF$ are also equilateral. Hence prove that DE and CF bisect each other.

6. Prove that the two angles adjacent to any one of the parallel sides of an isosceles trapezium are equal.

7. Prove that a parallelogram with a pair of supplementary opposite angles is a rectangle.

8. $ABCD$ is a parallelogram. Two squares ABPQ and $ADRS$ are drawn on the sides AB and AD respectively and are situated outside the parallelogram. Prove that $\triangle PRC$ is isosceles.

9. $\angle A$ of the parallelogram $ABCD$ is obtuse. Two equilateral triangles ABP and ADQ are situated outside the parallelogram. Prove that $\triangle CPQ$ is equilateral.

10. P is a point on the side BC of the square $ABCD$. Perpendicular drawn on AP from the point B intersects DC at the point Q. Prove that $AP = BQ$.

11. OP, OQ and OR are three line segments; three parallelograms $OPAQ$, $OQBR$ and $ORCP$ are drawn. Prove that AR, BP and CQ bisect each other.

12. $ABCD$ is a parallelogram and EF is a straight line outside it. Perpendiculars AP, BQ, CR and DS are drawn from the four-points A, B, C, D respectively lying on the line EF. Prove that $SP + SQ = SR$.

13. O is a point inside $\triangle PQR$. The parallelograms $QORX$,

ROPY and *POQZ* are drawn. Prove that $\triangle PQR \cong \triangle XYZ$.

14. *PQRS* is a parallelogram. The straight line drawn through the point *R* parallel to the bisector of $\angle Q$ meets *PQ* produced at the point *X* and *PS* produced at the point *Y*. Prove that $\triangle PXY$ is isosceles and the perimeter of the parallelogram is equal to the sum of *PX* and *PY*.

15. Through the points *A*, *B*, *C* of $\triangle ABC$ lines are drawn parallel to opposite sides. First and second parallel lines meet each other at the point *P*, second and third meet each other at the point *Q* and third and first meet each other at the point *R*. Prove that *B*, *C* and *A* are the mid-points of *OQ*, *QR* and *RP* respectively.

2.2 Conditions for a figure to be a parallelogram and concerned theorems.

The first condition of a quadrilateral to be a parallelogram is obtained from the definition. This condition is that if each pair of opposite sides of a quadrilateral are parallel then it is a parallelogram. Apart from this, there are some other conditions which are obtained from the following theorems.

Theorem 19

If the two pairs of opposite sides of a quadrilateral be equal, the quadilateral will be a parallelogram.

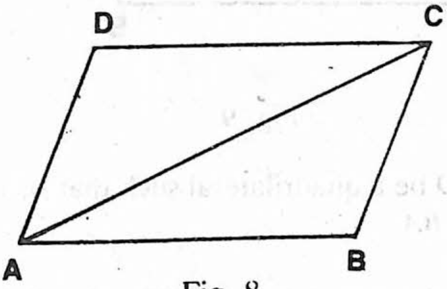


Fig. 8

Let $ABCD$ be a quadrilateral such that $AB=DC$ and $AD=BC$.

It is required to prove that $ABCD$ is a parallelogram.

Construction: Diagonal AC is drawn.

Proof: In $\triangle ABC$ and $\triangle CDA$

$$\left.\begin{array}{l} AB=DC \\ BC=AD \end{array}\right\} \text{(hypothesis)}$$

AC is common.

$\therefore \triangle ABC \cong \triangle CDA$

$\therefore \angle BAC = \angle ACD$ (corresponding)

But these are alternate angles.

$\therefore AB \parallel DC.$

Again, $\because \triangle ABC \cong \triangle ADC$

$\therefore \cong ACB = \cong CAD$ (corresponding)

But these are alternate angles.

$\therefore AD \parallel BC.$

$\therefore ABCD$ is a parallelogram.

Corollary : Rhombus is a parallelogram.

Theorem 20

If the opposite angles of a quadrilateral be equal, the quadrilateral will be a paralleogram.

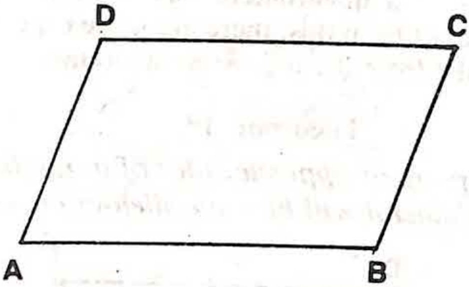


Fig. 9

Let $ABCD$ be a quadrilateral such that $\angle BAD = \angle DCB$ and $\angle ADC = \angle CBA$.

It is required to prove that $ABCD$ is a parallelogram.

Proof : $\angle BAD = \angle DCB$
and $\angle ADC = \angle CBA$ (hypothesis)

$\because \angle BAD + \angle ADC = \angle DCB + \angle CBA$
$= \frac{1}{2} (\angle BAD + \angle ADC + \angle DCB + \angle CBA$

$= \frac{1}{2} \times 4$ right angles ($\because$ sum of the four angles of a quadrilateral is 4 right angles)
$= 2$ right angles.

$\therefore AB \parallel DC$.

Similarly, it can be proved that $AB \parallel DC$.

$\therefore ABCD$ is a parallelogram.

Theorem 21

If any pair of opposite sides of a quadrilateral be equal and parallel, the quadrilateral will be a parallelogram.

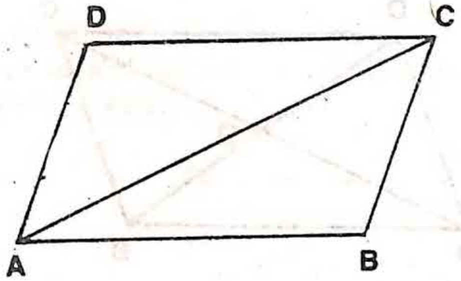


Fig. 10

Let $ABCD$ be a quadrilateral such that $AB=DC$ and $AB \parallel DC$

It is required to prove that $ABCD$ is a parallelogram.

Construction: Diagonal AC is drawn.

Proof: ∵ $AB \parallel DC$ and AC is a transversal
∴ $\angle BAC$ = alternate $\angle DCA$

Now, in $\triangle BAC$ and $\triangle DCA$
$AB=DC$ (hypothesis)
AC is common

and included $\angle BAC$ = included $\angle DCA$ (proved)

∴ $\triangle BCA \cong \triangle DCA$.
∴ $\angle BCA = \angle DAC$ (corresponding).

But these are alternate angles.
∴ $BC \parallel AD$.
∵ $AB \parallel DC$ (hypothesis)
and $BC \parallel AD$ (proved).

∴ $ABCD$ is a parallelogram.

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Theorem 22

If the diagonals of a quadrilateral bisect each other, the quadrilateral will be a parallelogram.

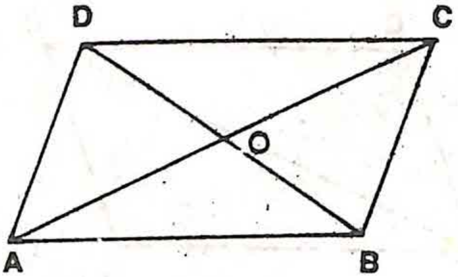


Fig. 11

Let the diagonals AC and BD of the quadrilateral $ABCD$ bisect each other at the point O, that is, $AO=OC$ and $BO=OD$.

It is required to prove that $ABCD$ is a parallelogram.

Proof: In $\triangle AOB$ and $\triangle COD$

$$\left.\begin{aligned} OA &= OC \\ OB &= OD \end{aligned}\right\} \text{(hypothesis)}$$

and included $\angle AOB$=included $\angle COD$ (vertically opposite)

$\therefore\ \triangle AOB \cong \triangle COD.$

$\therefore\ AB=DC$

and $\angle OAB=\angle OCD$

that is, $\angle CAB=\angle ACD$.

But these are alternate angles.

$\therefore\ AB \parallel DC.$

$\because\ AB=DC$ and $AB \parallel DC$

$\therefore\ ABCD$ is a parallelogram.

Remark: It can be proved that the sides BC and AD are parallel by proving that $\triangle BOC$ and $\triangle DOA$ are congruent. Then the two pairs of opposite sides will be parallel and hence from the definition it can be said that the quadrilateral $ABCD$ is a parallelogram.

Some applications :

1. Prove that, if a pair of opposite angles and a pair of opposite sides of a quadrilateral are equal,the quadrilateral will be a parallelogram.

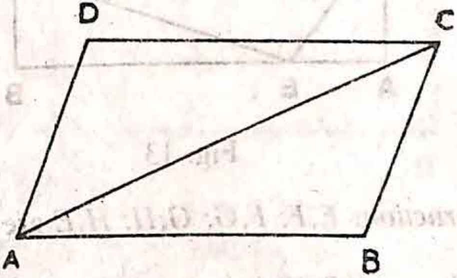


Fig. 12

Let $ABCD$ be a quadrilateral,such that $\angle ABC=\angle CDA$ and $AB \parallel DC$.

It is required to prove that $ABCD$ is a parallelogram.
Construction: Diagonal AC is drawn.

Proof: $\because$ $AB \parallel DC$ and AC is a transversal.
$\therefore$ $\angle BAC=$alternate $\angle DCA$.

Now,in $\triangle ABC$ and $\triangle ADC$
$\angle BAC=\angle DCA$ (proved)
$\angle ABC=\angle CDA$ (hypothesis)
$\therefore$ the remaining $\angle ACB$
=the remaining $\angle DCA$.

But these are alternate angles.
$\therefore$ $AD \parallel BC$.
$\because$ $AB \parallel DC$ and $AD \parallel BC$
$\therefore$ $ABCD$ is a parallelogram.

2. $ABCD$ is a rectangle. The points E, F, G, H are

situated on AB, BC, CD, DA respectively, such that $AE=BF=CG=DH$. Prove that $EFGH$ is a parallelogram.

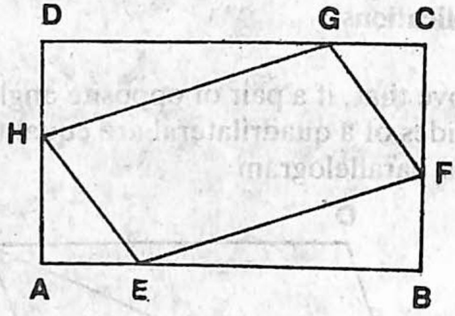


Fig. 13

Construction: E, F; F, G; G, H; H, E are joined.

Proof: $\because$ $ABCD$ is a rectangle $\therefore$ $AB=DC$

Again, $\because$ $AE=CG$ $\therefore$ $EB=DG$.

Similarly, $AH=FC$.

Now, in $\triangle GDH$ and $\triangle EBF$ ·

$DH=BF$ (hypothesis)
$DG=BE$ (proved)

and included $\angle GDH$ = included $\angle EBF$ [$\because$ each angle of a rectangle is a right angle]

$\therefore$ $\triangle GDH \cong \triangle EBF$.
$\therefore$ $HG=EF$.

Similarly, it can be proved that $EH=FG$.

$\therefore$ quadrilateral $EFGH$ is such that $EF=HG$ and $EH=FG$.

$\therefore$ $EFGH$ is a parallelogram.

3. If each of the three angles of a quadrilateral is a right angle, prove that the quadrilateral is a rectangle.

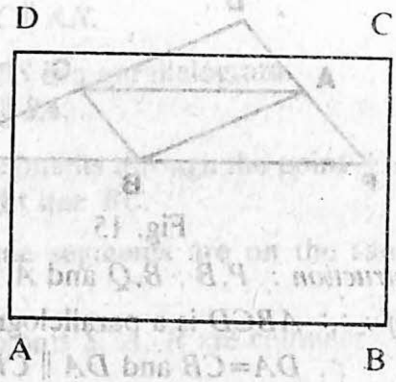


Fig. 14

Let each of $\angle A$, $\angle B$ and $\angle C$ of the quadrilateral $ABCD$ be one right angle.

It is required to prove that $ABCD$ is a rectangle.

Proof: $\angle A + \angle B + \angle C + \angle D = 4$ right angles

But $\angle A + \angle B + \angle C = 3$ right angles.

$\therefore$ $\angle D = 1$ right angle.

$\therefore$ $\angle A = \angle C$ and $\angle B = \angle D$ (each is a right angle)

$\therefore$ opposite angles of the quadrilateral $ABCD$ are equal.

$\therefore$ $ABCD$ is a parallelogram.

Again, $ABCD$ is a parallelogram and $\angle A$ is a right angle.

$\therefore$ $ABCD$ is a rectangle.

4. $ABCD$ is a parallelogram. DA and DC are produced to the points P and Q respectively, such that AP

= DA and $CQ = DC$. Prove that the three points P, B, Q are collinear.

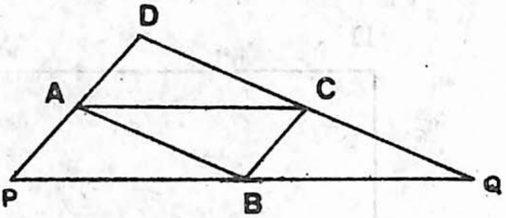


Fig. 15

Construction : P, B ; B, Q and A, C are joined.

Proof : $\because$ $ABCD$ is a parallelogram

$\therefore$ $DA = CB$ and $DA \parallel CB$;

But $AP = DA$ (Hypothesis)

$\therefore$ AP and CB are equal and parallel.

$\therefore$ APBC is a parallelogram.

$\therefore$ $AC \parallel PB$.

Similarly, AB and CQ are equal and parallel.

$\therefore$ $AC \parallel BQ$.

$\therefore$ two line segments through the point B are parallel to the same straight line AC.

$\therefore$ the two line segments are on the same straight line.

$\therefore$ the three points P, B, Q are collinear.

5. Two medians BP and CQ of $\triangle ABC$ are produced to the points R and S respectively, such that $BP = PR$ and $CQ = QS$. Prove that the three points S, A, R are collinear.

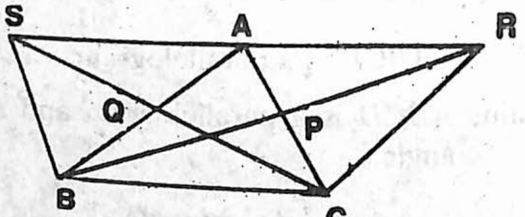


Fig. 16

Construction : S, A ; A, R ; S, B and R, C are joined.

Proof : ∵ BP is a median ∴ $AP=PC$.

Again, $BP=PR$ (hypothesis)

∴ $ABCR$ is a parallelogram.

∴ $BC \parallel AR$.

Similarly, $ACBS$ is a parallelogram.

$BC \parallel SA$.

∴ two line segments through the point A are parallel to the same straight line BC.

∴ the two line segments are on the same straight line.

∴ the three points S, A, R are collinear.

Exercise 5

1. *ABCD* is a parallelogram, *P* and *Q* are the mid-points of *AB* and *DC* respectively. Prove that APCQ is a parallelogram.

2. Diagonal *SQ* of the parallelogram *PQRS* is trisected at the points *K* and *L*. *PK* meets *SR* at the point *M* and *RL* meets *PQ* at the point *N*. Prove that *PMRN* is a parallelogram.

3. *ABCD* and *CDEF* are two parallelograms. Prove that *ABFE* is a parallelogram.

4. *ABCD* is a rectangle. On the two sides *AB* and *CD* two equilateral triangles *ABP* and CDQ are drawn on the outer side of the rectangle. Prove that *APCQ* is a parallelogram.

5. *KLMN* is a parallelogram. Equal lengths *KP*, *LQ*, *MR* and *NS* are cut off from *KL*, *LM*, *MN* and *NK* respectively. Prove that *PQRS* is a parallelogram.

6. *ABCD* is a parallelogram. *AD* is produced to the point *E*, such that $AD=DE$. *BE* intersects *CD* at the point *F*. Prove that $\triangle DEF \cong \triangle BFC$. Hence prove that the quadrilateral *BDEC* is a parallelogram.

7. Prove that the quadrilateral obtained by joining, in order, the mid-points of the sides of a parallelogram is another parallelogram.

8. Prove that the quadrilateral obtained by joining, in order, the mid-points of the sides of a rectangle is a rhombus.

9. Prove that the figure obtained by joining, in order, the mid-points of the sides of a rhombus is a rectangle.

10. Prove that the figure obtained by joining, in order, the mid-points of the sides of a square is another square.

11. *ABCD* is a quadrilateral. If $AD=BC$ and $\angle BAD=\angle ABC$, prove that *ABCD* is an isosceles trapezium.

12. *PQR* is a right-angled triangle with $\angle P$ right angle.

Mid-point of QR is A. ST is drawn through the point P parallel to QR in such a way that $PS=PT=PA$ and the points Q and S are on the same side of the line PA. Prove that $QRTS$ is a parallelogram.

13. PQR is a triangle. S and T are points on the sides PQ and PR respectively. Prove that RS and QT cannot bisect each other.

14. AC is a common diagonal of the two parallelograms $ABCD$ and $AECF$. If the points B, E, D; F are not collinear, prove that $BEDF$ is a parallelogram.

15. $ABCD$ is a quadrilateral. Two parallelograms $ABCE$ and $BADF$ are drawn. Prove that CD and EF bisect each other.

16. Prove that a parallelogram with equal diagonals is a rectangle. Hence prove that if AB and CD are two diameters of a circle, $ACBD$ is a rectangle.

17. PQRS is a rhombus. SA is drawn from the point S perpendicular to the line drawn through the point P parallel to QS. The point of intersection of the diagonals PR and QS is O. OA intersects PS at the point B. Prove that *(i)* $POSA$ is a rectangle, *(ii)* $BO=\frac{1}{2}PQ$.

2.3 Theorems on mid-points.

Theorem 23

The straight line drawn through the mid-point of one side of a triangle and parallel to a second side bisects the third side and the segment of this line between the sides of the triangle is half the second side.

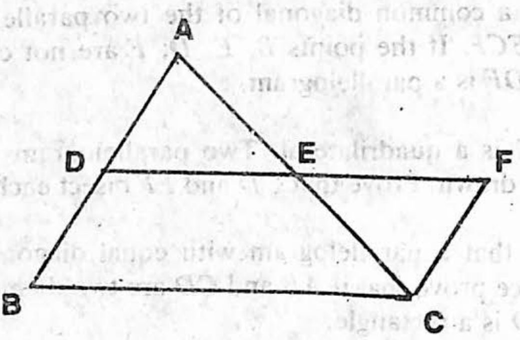


Fig. 17

Let us suppose that the straight line drawn through D, the mid-point of side AB of $\triangle ABC$ parallel to BC intersects AC at the point E.

It is required to prove that (i) $AE=EC$,

(ii) $DE=\frac{1}{2}BC$.

Construction : Through the point C a straight line is drawn parallel to BA to meet extended DE at the point F.

Proof : $\because$ $BD \parallel CF$ and $DF \parallel BC$

$\therefore$ $BCFD$ is a parallelogram.

$\therefore$ $BD=CF$.

But $BD=DA$ ($\because$ D is mid-point of AB), $\therefore$ $DA=CF$.

Now, in $\triangle ADE$ and $\triangle CFE$

$\angle DAE$=alternate $\angle FCE$ ($\because$ $BA \parallel CF$ and AC is a transversal)

$\angle AED$ = vertically opposite $\angle CEF$.

and $AD = CF$ (proved)

$\therefore \triangle ADE \cong \triangle CFE$

$\therefore AE = EC$ . . . . (i)

and $DE = EF$ (corresponding).

$\therefore DE = \frac{1}{2} DF = \frac{1}{2} BC$ . . . . (ii)

[*Remark* : This theorem can also be proved by constructions different from what has been done above

(*i*) Let us suppose that the straight line drawn through the point E parallel to AB intersects BC at the point F.

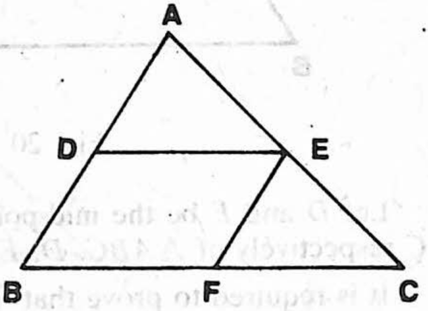


Fig. 18

Now prove the theorem after proving that $\triangle ADE$ and $\triangle EFC$ are congruent.

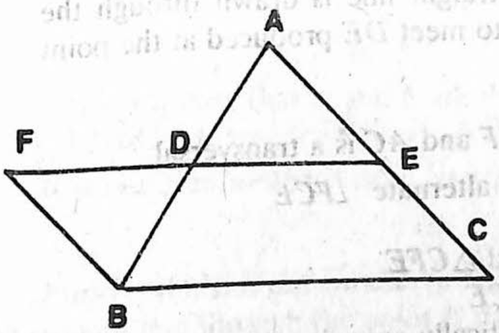


Fig. 19

(*ii*) ED is produced to the point F in such a way that $DF = ED$.

Now prove that $\triangle ADE$ and $\triangle BDF$ are congruent and $BCEF$ is a parallelogram.

Hence, prove the theorem.]

Theorem 24

The line-segment joining the mid-points of any two sides of a triangle is parallel to and half of the third side.

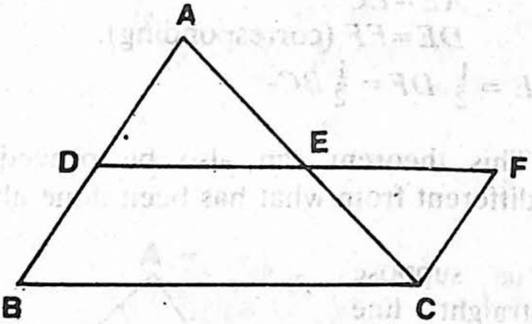


Fig. 20

Let D and E be the mid-points of the sides AB and AC respectively of $\triangle ABC$. D, E are joined.

It is required to prove that (i) $DE \parallel BC$,

(ii) $DE = \frac{1}{2} BC$.

Construction : A straight line is drawn through the point C parallel to BA to meet DE produced at the point F.

Proof : $\because$ $BA \parallel CF$ and AC is a transversal

$\therefore$ $\angle DAE$=alternate $\angle FCE$

Now, in $\triangle ADE$ and $\triangle CFE$

$\angle DAE = \angle FCE$

$\angle DEA$=vertically opposite $\angle FEC$

and $AE = EC$ ($\because$ E is the mid-point of AC)

These are corresponding sides.

$\therefore$ $\triangle ADE \cong \triangle CFE$

$\therefore$ $AD = FC$

But $AD=DB$ ($\because$ D is the mid-point of AB)

$\therefore$ $DB=FC$

$\because$ $DB \parallel FC$ and $DB=FC$

$\therefore$ $BCFD$ is a parallelogram

$\therefore$ $DF \parallel BC$

that is, $DE \parallel BC$ . . . . (i)

Again, $\because$ $\triangle ADE \cong \triangle CFE$

$\therefore$ $DE = EF = \frac{1}{2}\ DF = \frac{1}{2} BC$ . . . . (ii)

Alternative Proof :

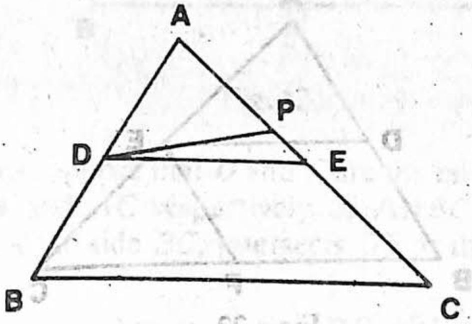


Fig. 21

Let us suppose that D and E are the mid-points of the sides AB and AC respectively. D, E are joined.

It is required to prove that (i) $DE \parallel BC$.

(ii) $DE = \frac{1}{2} BC$.

Proof : If DE is not parallel to BC, let us suppose that the straight line through the point D drawn parallel to BC intersects AC at the point P.

$\therefore$ P is the mid-point of AC.

But according to the hypothesis E is the mid-point of AC.

But it is impossible to have two mid-points of the same line segment.

Hence DE will be parallel to BC. . . . . (i)

$\because$ the straight line through the point D drawn parallel to BC intersects AC at the point E,

$\therefore$ $DE = \frac{1}{2} BC$. . . . . (ii)

[*Remarks* : (1) Since Theorem 24 is the converse of Theorem 23, this theorem is proved by the above method.

(2) In this case also, the theorem can be proved by means of different constructions.

(i) Through the point E a straight line is drawn parallel to AB to intersect BC at the point F.

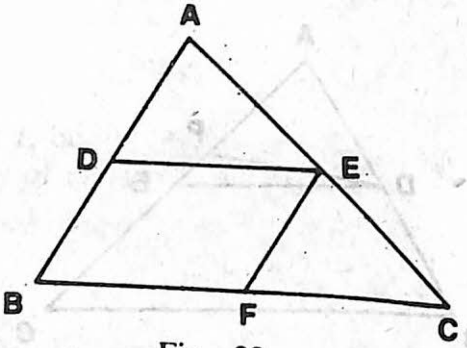


Fig. 22

With the help of Theorem 23 prove that $BDEF$ is a parallelogram. Hence prove the theorem.

(ii) DE is produced to the point F in such a way that $EF = DE$.

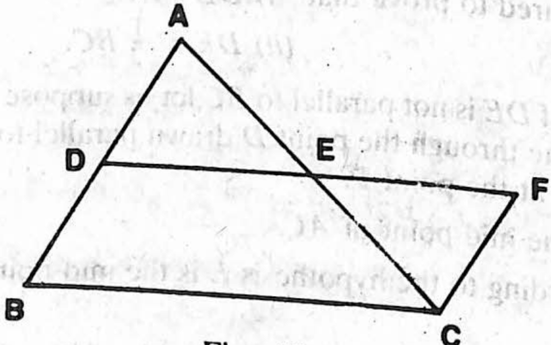


Fig. 23

Now prove the theorem by proving that $\triangle ADE$ and $\triangle CFE$ are congruent and that $BCFD$ is a parallelogram.]

Some Applications :

1. Prove that the line segment joining the mid-points of two sides of a triangle and the median on the third side bisect each other.

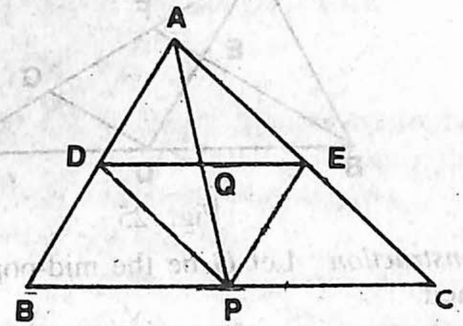


Fig. 24

Let us suppose that D and E are the mid-points of the sides AB and AC respectively of $\triangle ABC$ and AP, the median on the side BC, intersects DE at the point Q.

It is required to prove that $DQ=QE$ and $AQ=QP$.

Construction : P, D and P, E are joined.

Proof : $\because$ D and P are the mid-points of AB and BC

$\therefore$ $DP \parallel AC$, i.e., $DP \parallel AE$

and $DP=AE$ [$\because$ $DP=\frac{1}{2}AC$, $AE=\frac{1}{2}AC$]

$\therefore$ $DPEA$ is a parallelogram.

$\therefore$ DE and AP, the diagonals of the parallelogram, bisect each other.

$\therefore$ $DQ=QE$ and $AQ=QP$.

2. E is the mid-point of the median AD of $\triangle ABC$.

BE, when produced, intersects AC at the point F. Prove that $AF = \frac{1}{3} AC$.

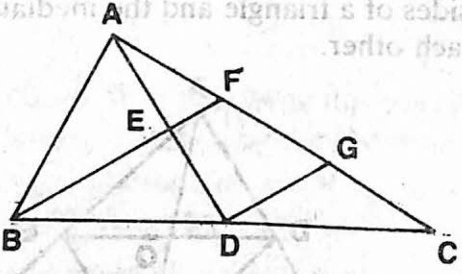


Fig. 25

Construction : Let G be the mid-point of FC. D, G are joined.

Proof : In $\triangle CBF$, D and G are the mid-points of the sides CB and CF respectively.

$\therefore DG \parallel BF$, that is, $DG \parallel EF$

Again, in $\triangle ADG$, E is the mid-point of AD and $EF \parallel DG$

$\therefore F$ is the mid-point of AG.

$\therefore AF = FG = GC$ [$\because$ by supposition G is the mid-point of FC.]

$\therefore AF = \frac{1}{3} AC$.

3. $ABCD$ is a square. Diagonals AC and BD intersect at the point O. The bisector of $\angle BAC$ intersects BO at the point P and BC at the point Q. Prove that $OP = \frac{1}{2} CQ$.

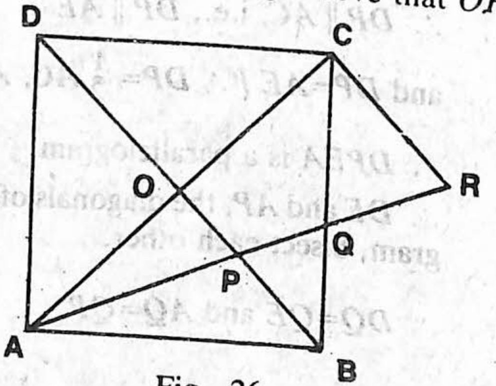


Fig. 26

Construction : Through the point C a straight line is drawn parallel to OP to intersect AQ produced at the point R.

Proof : $\because$ O is the mid-point of AC and $OP \parallel CR$
$\therefore$ from $\triangle ACR$, we get $OP=\frac{1}{2}\,CR$.

Again, $\because$ $OP \parallel CR$
and $\angle AOP=90°$ [$\because$ diagonals of a square bisect each other at right angles.]
$\therefore$ $\angle ACR=90°$ (corresponding angles)
$\therefore$ in $\triangle ACR$ and $\triangle ABQ$
$\therefore$ $\angle ACR = \angle ABQ$ [each is 90°]
$\angle CAR = \angle BAQ$ [AQ is the bisector of $\angle BAC$]
Remaining $\angle CRQ$=remaining $\angle AQB$.

But $\angle AQB = \angle CQR$ (vertically opposite)
$\therefore$ $\angle CRQ = \angle CQR$
in $\triangle CQR$, $CR=CQ$.
$\therefore$ $OP = \frac{1}{2}\,CQ$ [$\because$ $OP = \frac{1}{2}\,CR$]

4. E and F are mid-points of the oblique sides AD and BC respectively of the trapezium $ABCD$. Prove that $EF \parallel AB$ and $EF = \frac{1}{2}\,(AB+DC)$.

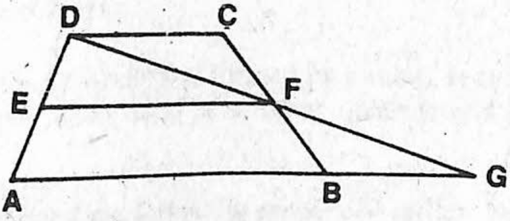


Fig. 27

Construction : D, F are joined and produced to meet AB produced at the point G.

Proof : In $\triangle CFD$ and $\triangle BFG$

$CF=BF$ [$\because$ F is the mid-point of BC]

$\angle CFD$=vertically opposite $\angle BFG$

$\angle CDF$=alternate $\angle BGF$ [$\because$ $DF \parallel BG$ and DG is a transversal]

$\therefore$ $\triangle CFD \cong \triangle BFG$

$\therefore$ $DF=FG$ (corresponding)

that is, F is the mid-point of DG.

Now, in $\triangle ADG$, E and F are the mid-points of the sides DA and DG respectively.

$\therefore$ $EF \parallel AG$

and $EF=\frac{1}{2} AG$

$=\frac{1}{2}(AB+BG)$

$=\frac{1}{2}(AB+DC)$ [$\because$ $\triangle CFD \cong \triangle BFG$]

Exercise 6

1. X, Y, Z are the mid-points of the sides AB, BC and CA respectively of $\triangle ABC$. Prove that $AXYZ$ is a parallelogram. Prove also that $\triangle AXZ$, $\triangle YZX$, $\triangle XBY$ and $\triangle ZCY$ are congruent.

2. D and E are points on the sides AB and AC respectively of $\triangle ABC$,such that $AD = \frac{1}{4} AB$ and $AE = \frac{1}{4}$ AC. Prove that $DE \parallel BC$ and $DE = \frac{1}{4} BC$.

3. D is the mid-point of the side BC of $\triangle ABC$. The straight lines drawn from D parallel to CA and BA intersect BA and CA at the points E and F respectively. Prove that $EF = \frac{1}{2} BC$.

4. A and B are the mid-points of the sides PQ and SR of the parallelogram $PQRS$. PB and AR intersect the diagonal SQ at the points C and D respectively. Prove that SQ is trisected at the points C and D.

5. X and Z are the mid-points of the sides QR and QP repectively of $\triangle PQR$. QP is produced to the point S in such a way that $PS=ZP$. SX intersects PR at the point Y. Prove that $PY = \frac{1}{4} PR$.

6. E and F are the mid-points of the sides AB and AC respectively of $\triangle ABC$. C, E and B, F are joined and produced to the points X and Y respectively in such a way that $EX=CE$ and $FY=BF$. Prove, with the help of Theorem 24,that the three points X, A, Y are collinear.

7. Prove that the quadrilateral formed by joining, in order, the mid-points of the four sides of another quadrilateral is a parallelogram.

8. You have proved the following properties earlier. Now, with the help of Theorem **24** :

(i) Prove that the quadrilateral formed by joining, in order, the mid-points of the sides of a parallelogram is a parallelogram.

4.

(ii) Prove that the quadrilateral formed by joining, in order, the mid-points of the sides of a rectangle is a rhombus.

(iii) Prove that the figure obtained by joining, in order, the mid-points of the sides of a rhombus is a rectangle.

(iv) Prove that the figure obtained by joining, in order, the mid-points of the sides of a square is a square.

9. $\angle A$ of $\triangle ABC$ is a right angle and D is the mid-point of the hypotenuse BC. Prove that $AD = \frac{1}{2} BC$.

10. E and F are the mid-points of the oblique sides AD and BC respectively of the trapezium $ABCD$. The line segment EF intersects the two diagonals AC and BD at the points G and H respectively. Prove that $EG=HF$. Hence prove that $2EF = AB+DC$.

11. In $\triangle ABC$, D and E are the mid-points of the sides AB and AC respectively. P and Q are the mid-points of CD and BD respectively. Prove that BE and PQ bisect each other.

12. The medians BE and CF of $\triangle ABC$ intersect at the point G. P and Q are the mid-points of BG and CG respectively. Prove that $PQEF$ is a parallelogram. Hence prove that each of the two medians is trisected at the point G.

13. The side AB of the parallelogram $ABCD$ is produced to the point E, such that $BE=AB$. DE intersects AC and BC at the points P and Q respectively. Prove that Q is the mid-point of BC and $PC = \frac{1}{2} AP$.

14. You have seen that Theorem 24 can be proved without taking help of Theorem 23. Hence we can also say that Theorem 23 is the converse of Theorem 24. Prove Theorem 23 as converse of Theorem 24.

2.4 Theorems on Transversals :

Theorem 25

If the lengths of the intercepts made by three or more parallel straight lines on a transversal are equal, the lengths of the intercepts made by them on any other transversal will also be equal.

First we prove the theorem for three parallel straight lines. It can then be proved for more than three paralllel straight lines.

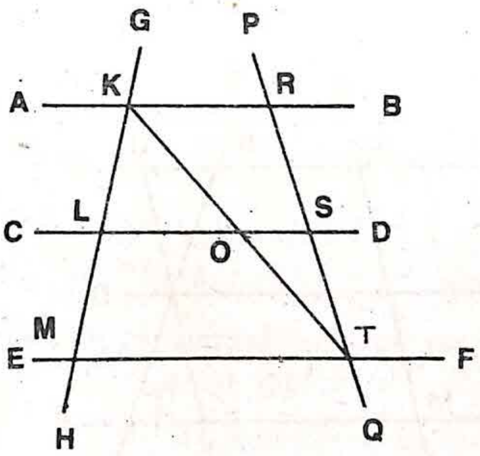


Fig. 28

Let us suppose that the intercepts KL, LM made by the three parallel straight iines AB, CD; EF on the transversal GH are equal, that is, $KL=LM$.

The intercepts made by those three parallel straight lines on the transversal PQ are RS and ST.

It is required to prove that $RS=ST$.

Construction : K, T are joined. Let KT intersect LS at the point O.

Proof : In $\triangle KMT$, L is the mid-point of KM (hypothesis)

and $LO \parallel MT$ (hypothesis)

$\therefore$ O is the mid-point of KT.

Again, in $\triangle TKR$, O is the mid-point of TK (proved) and $OS \parallel KR$ (hypothesis)

$\therefore$ S is the mid-point of RT.

That is, $RS=ST$.

Alternative Proof :

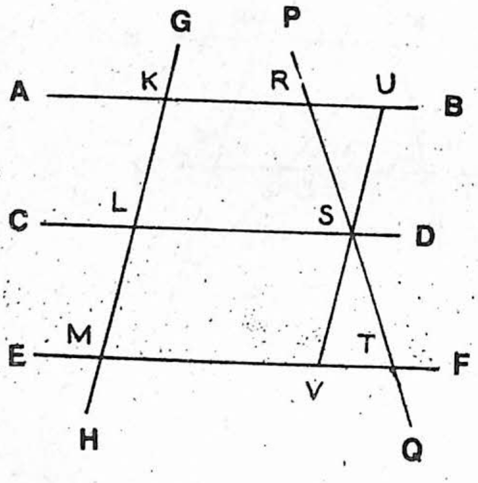


Fig. 29

Let us suppose that the intercepts KL, LM made by the three parallell straight lines AB, CD and EF on the transversal GH are equal. that is, $KL=LM$.

The intercepts made by those three parallel straight lines on another transversal PQ are RS and ST.

It is required to prove that $RS=ST$.

Construction : Through the point S a straight line is drawn parallel to GH to intersect AB and EF at the points U and V respectively.

Proof : In the quadrilateral $KLSU$
$KU \parallel LS$ (hypothesis)
$KL \parallel US$ (construction)
$\therefore$ $KLSU$ is a parallelogram
$\therefore$ $KL = US$.

Similarly, it can be proved that $LM = SV$.
But $KL = LM$
$\therefore$ $US = SV$

Now, from $\triangle SRU$ and $\triangle SVT$
$\angle URS$=alternate $\angle VTS$ [$\because$ $AB \parallel EF$ and PQ is a transversal]
$\angle USR$=vertically opposite $\angle VST$
$SU = SV$ (proved)

and these are corresponding sides of the two triangles.
$\therefore$ $\triangle SRU \cong \triangle SVT$
$\therefore$ $SR = ST$ (corresponding), i.e., $RS = ST$.

[*Remarks :* (1) Note that in the first proof we have taken help of Theorem 23, whereas in the second proof we have not taken its help.

(2) Supposing that the property is proved for three parallel straight lines, prove the property for four or more parallel lines.]

Corollary : 1. One side of a triangle is divided into a number of equal parts. From the points of division straight lines are drawn parallel to one of the other two sides. These parallel lines will divide the third side of the triangle into as many equal parts as the first side.

2. A straight line through the mid-point of an oblique side of a trapezium drawn parallel to its two parallel sides will bisect the other oblique side.

Some applications :

1. In $\triangle ABC$, D is the mid-point of BC. PQ is any straight line through the point A. Perpendiculars drawn from the points B, C and D to PQ intersect it at the points L, M and N respectively. Prove that $DL = DM$.

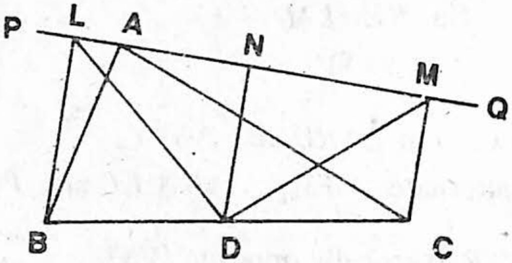


Fig. 30

Proof : $\because$ each of BL, DN and CM is perpendicular to PQ ; therefore, they are parallel to each other.

$\because BL \parallel DN \parallel CM$

and $BD = DC$

$\therefore LN = NM$

Now, in $\triangle DLN$ and $\triangle DMN$

$LN = NM$ (proved)

DN is common to both

and included $\angle DNL$ = included $\angle DNM$

[$\because DN \perp PQ$]

$\therefore \triangle DLN \cong \triangle DMN$

$\therefore DL = DM$ (corresponding).

Exercise 7

1. C is the mid-point of AB, and PQ is any striaght line. Least distances of the points A, B and C from PQ are AR, BS and CT respectively. Prove that $AR+BS=2CT$.

2. AD is a median of $\triangle ABC$. Two straight lines BE and CF are parallel to DA. A straight line perpendicular to AD intersects BE and CF at the points P and Q respectively. Prove that $AP = AQ$.

3. In a trapezium $ABCD$, the straight line through the mid-point of an oblique side AD and parallel to AB intersects BC and the two diagonals AC and BD at the points F, G and H respectively. Prove that $EG=HF$ and $EF = \frac{1}{2}(AB+DC)$.

4. $ABCD$ is a parallelogram and PQ is a straight line outside the parallelogram. AE, BF, CG, DH are perpendiculars to PQ from the points A, B, C, D respectively. OR is the perpendicular from O, the point of intersection of the two diagonals of the parallelogram, to PQ. Prove that $AE + CG = 2OR = BF + DH$.

5. In Class VIII you have learnt two methods of dividing a straight line into a number of equal parts. Take a line segment AB and divide it into six equal parts by the above two methods.

6. You have seen that Theorem 25 can be proved without the help of Theorem 23. Now prove Theorem 23 with the help of Theorem 25.

2.5 **Area :**

You have already some idea about the area of a plane figure. For having better idea it is necessary to have knowledge about some other concepts. These concepts are *region* of a plane figure and the *boundary* and *inside* of that region.

The collection of all the points that are inside and on a plane figure is called its *region*. The plane figure is called its *boundary* and the collection of points within the plane figure is called its *inside*.

If we suppose that the square region of a definite measure is associated with the number '1', then that square may be called a unit square. Now let us consider a rectangle whose length and breadth are respectively equal to 5 times and 3 times the length of a side of the above square. Then the unit square may be placed 15 times in the region of the rectangle. That is, with respect to the square it may be supposed that the region of the rectangle is related with the number '15'. This number '15' may be said to be the area of the rectangle.

Similarly, any plane figure may be taken to be related to a positive number. The number related to the region of the above figure will be called the area of the plane figure with respect to the square. Remember that area is a magnitude. You have also learnt to calculate areas of rectangle and square. Now we shall discuss relationship between the areas of some plane figures like parallelogram, rectangle and triangle. Before doing so, let us have some more discussion about area.

We have already said that area is a magnitude. But there is a dissimilarity between the ideas of the value of area and the magnitudes of the length of a line segment and of an angle. If the lengths of two line segments be equal, one of them can be made to coincide with the other. This is also true for two angles. But this may not be true for area. If the areas of two figures be equal it may not be possible to make one of them coincide with the other: that is, the two figures may not be congruent; the two figures may not even be of same geometric character.

For example, the areas of a rectangle of length 9 cm and breadth 4 cm, a rectangle of length 12 cm and breadth 3 cm and a square of side 6 cm are same; even though, inspite of the first two being rectangles they are not congruent and the third is a different type of geometric figure. But two congruent figures will always have same area.

In proving the thoerems we shall assume that

(i) congruent figures are of equal area,

(ii) if the area of a figure is added to each of the areas of two figures of equal areas the sums will also be equal,

(iii) if the areas of two figures of equal area be added to the areas of two other figures of equal area, the sums will also be equal,

(iv) if the areas of two figures of equal area are subtracted from the area of a figure, the differences will be equal,

(v) if the area of a figure is subtracted from the areas of two figures of equal area, the differences will be equal,

(vi) if the areas of two figures of equal area be subtracted from the areas of two other figures of equal area, the differences will be equal,

(vii) if a figure is divided into two parts, the sum of the areas of the two parts will be equal to the area of the whole figure,

(viii) if one figure is a part of the other, the area of the part will be less than the area of the whole figure.

We shall discuss some more definitions before discussing theorems on area.

1. Altitude of a parallelogram :

The perpendicular distance between any pair of parallel sides of a parallelogram will be called the altitude of the parallelogram. In this case, any one of those two sides will be taken as the base.

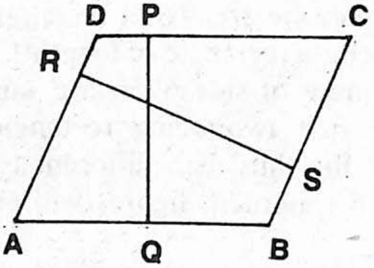


Fig. 31

In fig. 31, if AB or CD be taken as the base of the parallelogram $ABCD$, then PQ will be its altitude, whereas if any one of AD or BC be taken as its base, the altitude will be RS.

2. Altitude of a triangle :

The distance between any side of the triangle and the opposite vertex will be called the *altitude* of the triangle. That is, the length of the perpendicular drawn from any vertex of the triangle to its opposite side will be called its altitude. The opposite side will be called base. Any one side of the triangle may be taken as base.

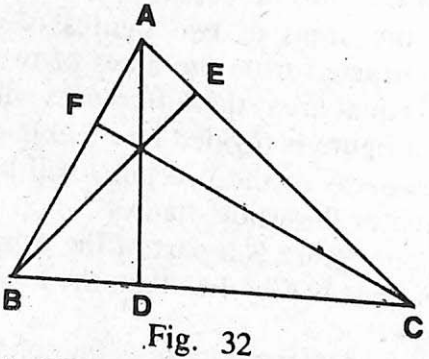


Fig. 32

In Fig. 32, AD is the altitude when BC is base, BE is the altitude when AC is base and CF is the altitude when AB is base.

Note further that, in the following figure, the parallelograms and the triangles lie between the same pair of parallel lines.

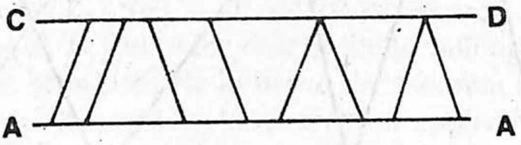


Fig. 33

In this case, the altitudes of the parallelograms and the triangles are same (if the sides lying on one of the parallel lines are taken as bases). This altitude is equal to the distance between the two parallel lines.

2.6 Theorems on area :

Theorem 26

Parallelograms on the same base and between the same parallels are equal in area.

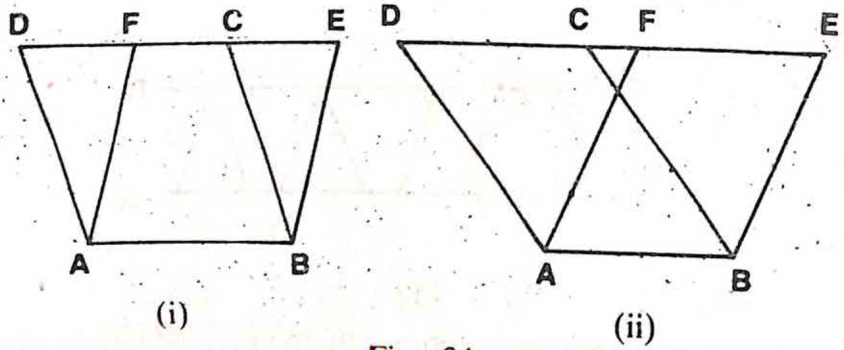


Fig. 34

Let the parallelograms $ABCD$ and $ABEF$ be on the same base AB and between the same parallels AB and DE.

It is required to prove that

parallelogram $ABCD$ = parallelogram $ABEF$.

Proof : AD and BC are opposite sides of the parallelogram $ABCD$.

$\therefore$ AD and BC are parallel and DE is a transversal.
$\therefore$ $\angle BCE$ = corresponding $\angle ADF$.

Again, AF and BE are opposite sides of the parallelogram $ABEF$

$\therefore$ AF and BE are parallel and DE is a transversal,
$\therefore$ $\angle BEC$ = corresponding $\angle AFD$.

Now, in $\triangle BEC$ and $\triangle AFD$

$\angle BCE = \angle ADF$ (proved)
$\angle BEC = \angle AFD$ (proved)
$BE = AF$ (opposite sides of a parallelogram)

$\therefore \triangle BEC \cong \triangle AFD$

$\therefore \triangle BEC = \triangle AFD$

$\therefore$ quadrilateral $ABED - \triangle BEC$
$= \text{quadrilateral } ABED - \triangle AFD.$

$\therefore$ parallelogram $ABCD$ = parallelogram $ABEF$.

[*Remark :* In the case of the first figure,the areas of the two parallelograms could also be proved as equal by adding the areas of $\triangle BEC$ and $\triangle AFD$ to the area of the quadrilateral $ABCF$. But in the case of the second figure,it could not have been possible to prove the theorem in the above manner. That is why, as a method applicable to both cases, the areas of the two triangles have been subtracted from the area of the whole quadrilateral.]

Corollary : 1. Parallelograms with same base and same altitude are equal in area.

Parallelograms with same altitude will lie between same parallels.

2. A rectangle and a parallelogram on the same base and between the same parallels are equal in area.

3. Area of a parallelogram = base × altitude.

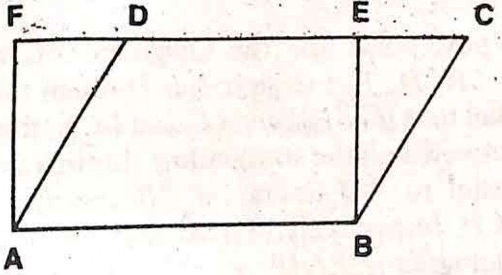


Fig. 35

Parallelogram $ABCD$ = rectangle $ABEF$.
But, area of the rectangle $ABEF = AB.BE$.

$\therefore$ area of the parallelogram $ABCD = AB.BE$.

But with respect to the base AB, BE is the altitude of the parallelogram.

$\therefore$ area of a parallelogram = base × altitude.

4. If two parallelograms have equal bases and lie between same parallels, they are equal in area.

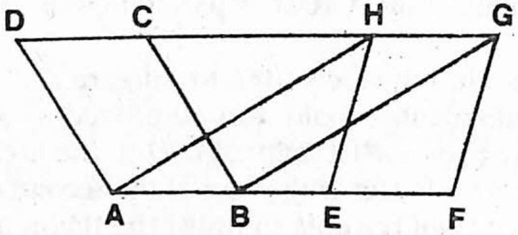


Fig. 36

Two parallelograms $ABCD$ and $EFGH$ have equal bases AB and EF and lie between the same parallels AF and DG.

It is required to prove that they are equal in area.

Proof : Let the distance between the two parallel lines, i.e., the altitude of the two parallelograms be h.

$$\begin{aligned}\text{Then } \square ABCD &= AB \times h\\ &= EF \times h \ [\because AB = EF]\\ &= \square EFGH.\end{aligned}$$

Some Applications :

1. O is any point on the diagonal AC of the parallelogram $ABCD$. The straight line through the point O drawn parallel to AB intersects AD and BC at the points E and F respectively and the straight line through the point O drawn parallel to AD intersects AB and DC at the points G and H respectively. Prove that parallelogram $BGOF$=parallelogram $DEOH$.

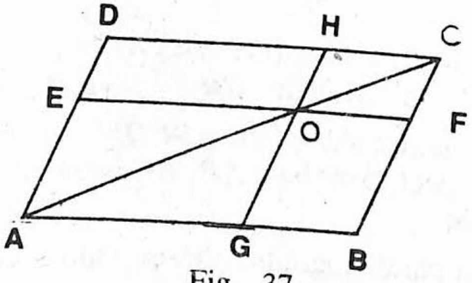


Fig. 37

Proof : $EF \parallel AB \parallel DC$ and $AD \parallel GH \parallel BC$.

$\therefore$ each of $AGOE$, $BGOF$, $CFOH$ and $DEOH$ is a parallelogram.

AC is a diagonal of the parallelogram $ABCD$.

$$\therefore \triangle ABC \cong \triangle ADC$$

$$\therefore \triangle ABC = \triangle ADC$$

Similarly, $\triangle AGO = \triangle AEO$ and $\triangle OFC = \triangle OHC$

$$\therefore \triangle ABC - \triangle \text{f48AEO} - \triangle OFC$$

$$= \triangle ADC - \triangle AEO - \triangle OHC$$

$$\therefore \square BGOF = \square DEOH.$$

[*Remarks* : (1) Parallellograms $AGOE$ and $CFOH$ are called parallelograms about the diagonal AC and the parallelograms $BGOF$ and $DEOH$ are called their respective complements.

From this we can enunciate the above problem in the following way :

The complements of the two parallelograms which are about the diagonal of a parallelogram are equal in area.

(2) Using this property we can construct a parallelogram equal in area to a given parallelogram if one of the sides of the required parallelogram be given.

Let $ABCD$ be a parallelogram. It is required to construct a parallelogram equal in area to the given parallelogram when the length of a side of the required parallelogram is equal to the length of a line segment p.

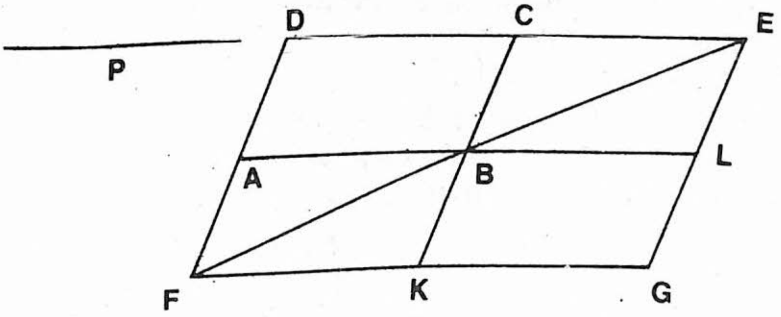


Fig. 38

Construction : DC is produced to the point E such that $CE=p$. E,B are joined and produced to meet DA produced at the point F. The parallelogram $FDEG$ is formed. AB and CB produced intersect GE and FG at the points L and K respectively.

Then $BKGL$ is the required parallelogram.]

2. Two parallelograms $ABCD$ and $ABEF$ are on the same side of AB and the area of $ABEF$ is greater than that of $ABCD$. Prove that

(i) $DCEF$ is a parallelogram

(ii) $\square$ ABEF − $\square$ ABCD = $\square$ DCEF

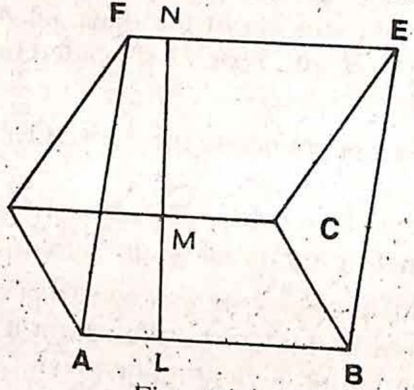


Fig. 39

Proof: (i) EF = and $\parallel$ AB ($\because$ $ABEF$ is a parallelogram)

Also, EF = and $\parallel$ DC ($\because$ ABCD is a parallelogram)

$\therefore$ $DCEF$ is a parallelogram.

(ii) Let a perpendicular transversal to the three parallel lines AB, DC and FE intersect them at the points L, M and N respectively.

Then the altitudes of the parallelograms $ABCD$, $ABEF$ and $DCEF$ are LM, LN and MN respectively.

$$\therefore \square ABEF - \square ABCD = AB.LN - AB.LM$$
$$= AB(LN - LM)$$
$$= AB.MN$$
$$= DC.MN$$

= base × altitude of the parallelogram $DCEF$
= $\square$ $DCEF$.

Exercise 8

1. In the parallelogram $ABCD$, P and Q are mid-points of AB and DC. Prove that $\square\, APCQ = \frac{1}{2} \square\, ABCD$.

2. (i) P is a point on the side AD of the parallelogram $ABCD$. The straight line through the point C drawn parallel to BP intersects AD produced at the point Q and the straight line through the point Q drawn parallel to PC intersects BC produced at the point R. Prove that $CPQR$ is a parallelogram and its area is equal to that of $ABCD$.

(ii) In the above figure, if RQ produced intersects BP produced at the point S, prove that $\square\, CPSQ = \square\, ABCD$.

3. In $\triangle ABC$, D, E and F are the mid-points of the sides BC, CA and AB respectively. In $\triangle DEF$, K, L, M are the mid-points of EF, FD, DE respectively. KL and KM intersect BC at the points P and Q respectively. Prove that quadrilateral $CEKQ$=quadrilateral $BFKP$.

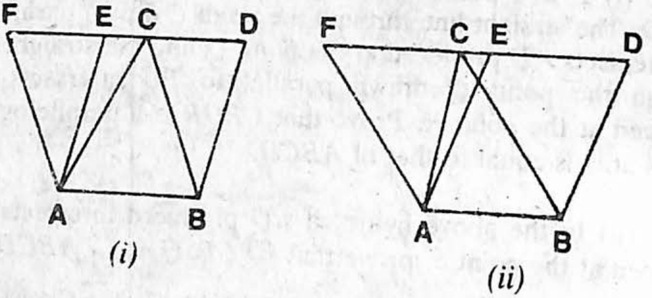
4. Two parallelograms $ABCD$ and $ABEF$ are on the opposite sides of AB in such a way that the points D, A, F are non-colliner. Prove that $DCEF$ is a parallelogram and $\square\, ABCD + \square\, ABEF = \square\, DCEF$.

5. In the rhombus $ABCD$, the distance between the sides AB and DC is PQ and that between AD and BC is RS. Prove that $PQ = RS$.

6. Prove that the area of the sqaure $ABCD$ is more than that of the rhombus $ABEF$.

Theorem 27

If a triangle and a parallelogram are on the same base and between the same parallels, the area of the triangle is half that of the parallelogram.



(i) (ii)

Fig. 40

Let $\triangle ABC$ and parallelogram $ABDE$ be on the same base AB and between the same parallels AB and ED [Fig. 40 (i)] or AB and CD [Fig. 40 (ii)].

It is required to prove that $\triangle ABC = \frac{1}{2} \square ABDE$.

Construction : The straight line through the point A drawn parallel to BC intersects DC produced at the point F.

Proof : By construction $ABCF$ is a parallelogram and AC is one of its diagonals.

$$\therefore \triangle ABC = \frac{1}{2} \square \text{ABCF}$$

$$\therefore \triangle ABC = \frac{1}{2} \square \text{ABDE}$$

[$\because$ the two parallelograms are on the same base and between the same parallels.]

[*Remark :* The theorem could also be proved by drawing a straight line through the point C parallel to either EA or DB. But in these two cases, the reasonings would have been a little different. Try to prove the theorem in this way.]

Corollaries :

1. If a triangle and a parallelogram are on the same base and have same altitude, the area of the triangle will be half that of the parallelogram.

If they have same altitude, they will be between the same parallels. Hence the area of the triangle will be equal to half that of the parallelogram.

2. If a triangle and a rectangle be on the same base and between the same parallels, the area of the triangle will be half that of the rectangle.

3. Area of a triangle = $\frac{1}{2}$ × base × altitude.

$\triangle ABC$ and rectangle $BCDE$ are on the same base BC and between the same parallels BC and ED.

$$\therefore \triangle ABC = \tfrac{1}{2} \text{ rectangle } BCDE = \tfrac{1}{2}\, BC.CD$$
$$= \tfrac{1}{2}\, BC.AP \; (\because APCD \text{ is a rectangle.})$$

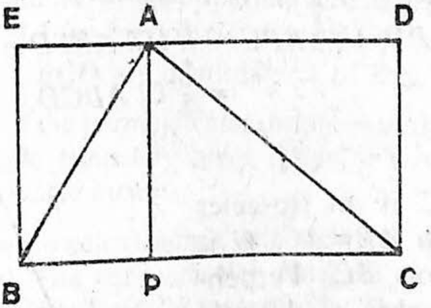


Fig. 41

4. If a triangle and a parallelogram have equal bases and lie between the same parallels, the area of the triangle will be equal to half that of the parallelogram.

Some Applications :

1. P is any point inside the parallelogram $ABCD$. Prove that $\triangle APD + \triangle BPC = \frac{1}{2} \square$ ABCD.

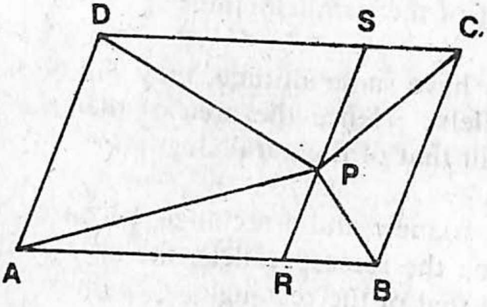


Fig. 42

Construction : Through the point P a straight line is drawn parallel to BC to intersect the sides AB and DC at the points R and S respectively.

Proof : $\triangle APD = \frac{1}{2} \square ARSD$

[$\because$ they have the same base AD and lie between the same parallels AD and RS.]

Similarly, $\triangle BPC = \frac{1}{2} \square BRSC$

$$\therefore \triangle APD + \triangle BPC = \frac{1}{2} (\square ARSD + \square BRSC)$$
$$= \frac{1}{2} \square ABCD.$$

2. ABC is an isosceles triangle with $AB = AC$. O is any point on BC. Perpendicular distances of AB and AC from the point O are OP and OQ respectively and that of AC from the point B is BD. Prove that $OP + OQ = BD$.

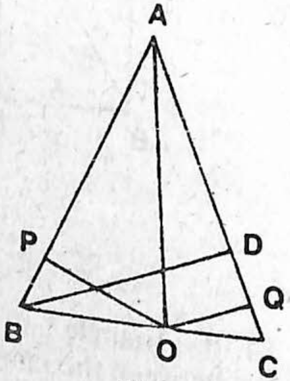


Fig. 43

Construction : A, O are joined.

Proof : $\triangle AOB = \frac{1}{2} AB.OP = \frac{1}{2} AC.OP$

$[\because AB = AC]$

$\triangle AOC = \frac{1}{2} AC.OQ$

$\therefore \triangle AOB + \triangle AOC = \frac{1}{2} AC.(OP+OQ)$

Again, $\triangle ABC = \frac{1}{2} AC.BD$

$\because \triangle ABC = \triangle AOB + \triangle AOC$

$\therefore \frac{1}{2} AC.BD = \frac{1}{2} AC.(OP+OQ)$

$\therefore BD = OP + OQ.$

[*Remark* : Try to prove this without using the formula for the area of a triangle and by cutting DE from DB equal in length to PO.]

Exercise 9

1. In the parallelogram $ABCD$, P and Q are mid-points of AB and DC respectively. Prove that $PBQD$ is a parallelogram and $\triangle PBC = \frac{1}{2} \square PBQD$.

2. Side AB and DC of the trapezium $ABCD$ are parallel and $AB > DC$. From BA the portion BE is cut off equal in length to CD. Prove that $\triangle ACD = \frac{1}{2}$ quadrilateral $BCDE$.

3. The sum of the perpendicular distances of the sides of an equilateral triangle from any point inside it is equal to the altitude of the triangle prove.

4. ABC is an isosceles triangle with $AB = AC$. P is any point on BC produced. The perpendicular distances of AB and AC from the point P are PQ and PR respectively. The perpendicular distance of AC from the point B is BS. Prove that $PQ - PR = BS$.

5. ABC is an equilateral triangle. O is a point outside the triangle and within the angular region ABC. OP, OQ and OR are the perpendicular distances of AB, BC and CA respectively from the point O. Prove that the altitude of the triangle $= OP + OQ - OR$.

6. In the parallelogram $ABCD$ a straight line parallel to the side AB intersects AD, AC and BC or those line segment produced at E, F and G respectively. Prove $\triangle AEG = \triangle AFD$.

7. E is any point on the side DC of the parallelogram $ABCD$. AE produced intersects BC produced at F. D, F are joined. Prove that (i) $\triangle ADF = \triangle ABE$ (ii) $\triangle DEF = \triangle BEC$.

8. O is a point outside the parallelogram $ABCD$ and is in the region bounded by AB produced and DC produced. Prove that

(i) $\triangle AOD - \triangle BOC = \frac{1}{2} \square ABCD$

(ii) $\triangle AOD = \triangle AOB + \triangle COD + \triangle BOC$.

Theorem 28

Triangles on the same base and between the same parallels are equal in area.

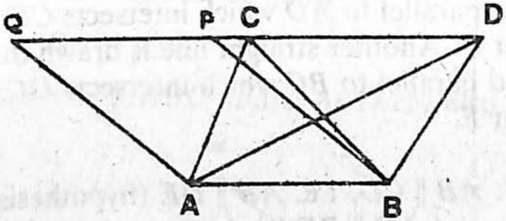


Fig. 44

Let $\triangle ABC$ and $\triangle ABD$ be on the same base AB and between the same parallels AB and CD.

It it required to prove that $\triangle ABC = \triangle ABD$.

Construction : A parallelogram $ABPQ$ is constructed with AB as base and lying between the parallels AB and CD.

Proof : $\because$ $\triangle ABC$ and $\square$ $ABPQ$ are on the same base AB and between the same parallels AB and QD,

$\therefore$ $\triangle ABC = \frac{1}{2} \square \text{ABPQ}$

Similarly, $\triangle ABD = \frac{1}{2} \square ABPQ$

$\therefore$ $\triangle ABC = \triangle ABD$.

[*Remark :* Since the relationship between the areas of a triangle and a parallelogram on the same base and between the same parallels is known to us. so the parallelogram $ABPQ$ is constructed.]

Alternative Proof :

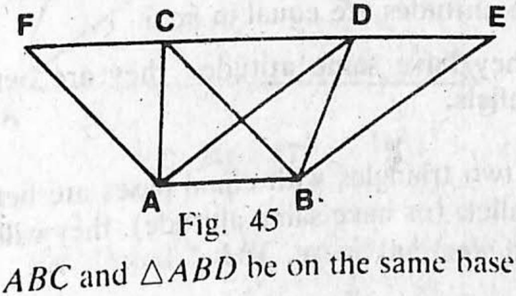


Fig. 45

Let $\triangle ABC$ and $\triangle ABD$ be on the same base AB and between the same parallels AB and CD.

It is required to prove that $\triangle ABC = \triangle ABD$.

Construction : A straight line is drawn through the point B and parallel to AD which intersects CD produced at the point E. Another straight line is drawn through the point A and parallel to BC which intersects DC produced at the point F.

Proof : $AB \parallel CD$, i.e. $AB \parallel DE$ (hypothesis)
and $AD \parallel BE$ (by construction)

$\therefore$ $ABED$ is a parallelogram.

Similarly, $ABCF$ is also a parallelogram.

$$\therefore \quad \square ABED = \square ABCF$$

Again, BD is a diagonal of the parallelogram $ABED$.

$$\therefore \triangle ABD = \frac{1}{2} \square ABED$$

Similarly, $\triangle ABC = \frac{1}{2} \square ABCF$

$$\therefore \triangle ABC = \triangle ABD$$

$$(\because \square ABED = \square \quad ABCF)$$

[*Remarks* : (1) In the proof, help of Theorem 27 is not taken.

(2) The parallelograms are consructed because we have seen earlier that diagonal divides a parallelogram in two equal parts and that two parallelograms on the same base and between the same parallels are equal in area.]

Corollaries : 1. Two triangles on the same base and with same altitudes are equal in areu.

As they have same atitude, they are between the same parallels.

2. If two triangles with equal bases are between the same parallels (or have same altitude), they will be equal in area.

3. Median of a triangle divides it into two triangles of equal areas.

AD is a median of $\triangle ABC$.

$\therefore\ BD = DC$.

Altitudes of $\triangle ABD$ and $\triangle ACD$ are also equal.

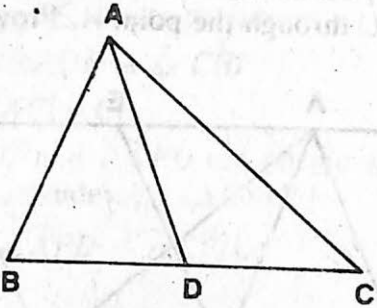


Fig. 46

$\therefore\ \triangle ABD$ and $\triangle ACD$ are on equal bases and have same altitude.

$\therefore\ \triangle ABD = \triangle ACD$.

Some applications :

1. The sides AB and DC of a trapezium $ABCD$ are parallel. Diagonals AC and BD intersect each other at the point O. Prove that $\triangle AOD = \triangle BOC$.

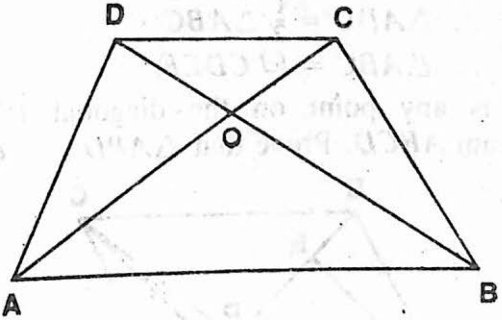


Fig. 47

Proof : $\triangle ABD$ and $\triangle ABC$ are on the same base AB

and between the same parallels AB and DC.

$\therefore \triangle ABD = \triangle ABC$

$\therefore \triangle ABD - \triangle AOB = \triangle ABC - \triangle AOB$

$\therefore \triangle AOD = \triangle BOC$

2. D is the mid-point of the side BC of $\triangle ABC$. Parallelogram $CDEF$ lies between BC and the straight line parallel to BC through the point A. Prove that $\triangle ABC = \square\, CDEF$.

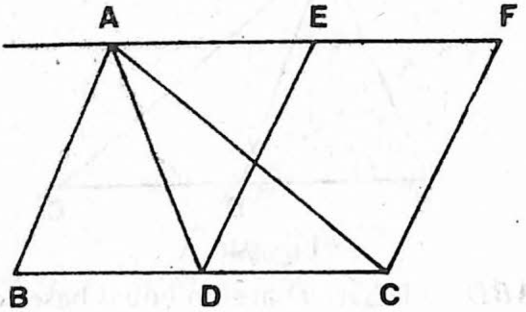


Fig. 48

Construction : A, D are joined.

Proof : $\because$ $\triangle ADC$ and $\square\, CDEF$ are on the same base DC and between the same parallels DC and AF

$\therefore \triangle ADC = \frac{1}{2} \square\, CDEF$

Again, $\because$ AD is a median of $\triangle ABC$

$\therefore \triangle ADC = \frac{1}{2} \triangle ABC$

$\therefore \triangle ABC = \square\, CDEF$.

3. P is any point on the diagonal BD of the parallelogram $ABCD$. Prove that $\triangle APD = \triangle CPD$.

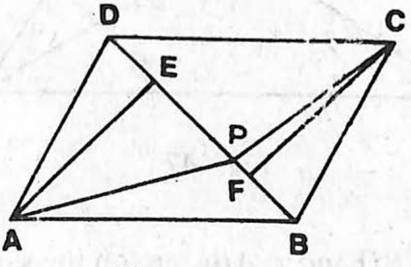


Fig. 49

Construction : Perpendiculars are drawn from the points A and C to BD to intersect it at the points E and F respectively.

Proof : In $\triangle ADE$ and $\triangle CBF$

$\angle ADE$ = alternate $\angle CBF$ [$\because$ AD BC]

$\angle AED = \angle CFB$ [each is a right angle]

$AD=BC$ [opposite sides of a parallelogram]

$\therefore \triangle ADE \cong \triangle CBF$

$\therefore AE = CF.$

Now $\triangle APD$ and $\triangle CPD$ are on the same base DP and have equal altitudes. [$\because$ $AE=CF$]:

$\therefore \triangle APD = \triangle CPD.$

[*Remark :* We could also prove this by drawing the diagonal AC instead of drawing the perpendiculars. Try to prove in this way.]

Exercise 10

1. P is any point on the median AD of the triangle ABC. Prove that $\triangle ABP = \triangle ACP$.

2. AD and BE are two medians of the triangle ABC. Prove that $\triangle ACD = \triangle BCE$.

3. A striaght line parallel to the side BC of the triangle ABC intersects AB and AC at the points P and Q respectively. CP and BQ intersect each other at the point X. Prove that *(i)* $\triangle BPQ = \triangle CPQ$, *(ii)* $\triangle BCP = \triangle BCQ$, *(iii)* $\triangle ACP$, *(iv)* $\triangle BXP = \triangle CXQ$..

4. Prove that the two diagonals of a parallelogram divide the parallelogram into four triangles of equal areas.

5. $ABCD$ is a quadrilateral. The straight line through the point C parallel to the diagonal DB intersects AB produced at the point E. Prove that quadrilateral $ABCD = \triangle ADE$.

6. P is the mid-point of the diagonal BD of the quadrilateral $ABCD$. Prove that the area of the quadrilateral formed by joining A, P and C, P $\frac{1}{2}$ quadrilateral $ABCD$.

7. D is the mid-point of the side BC of $\triangle ABC$ and P is any point on the side P, A are joined. The straight line through the point D parallel to PA intersects AB at the point Q. $\triangle ADQ = \triangle PDQ$. Hence, prove that $\triangle BPQ = \frac{1}{2} \triangle ABC$.

Theorem 29

Triangles of equal areas standing on the same base and on the same side of it are between the same parallels.

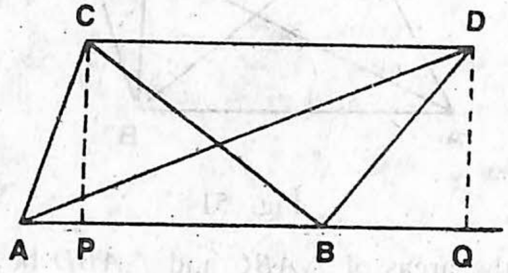


Fig. 50

Let the areas of $\triangle ABC$ and $\triangle ABD$ be equal and they lie on the same side of the common base AB.

C, D are joined.

It is required to prove that $AB \parallel CD$.

Construction : CP and DQ are drawn perpendiculars from the points C and D respectively to AB or AB produced.

Proof : $\triangle ABC = \frac{1}{2} AB.CP$ [$\because$ AB is base and CP is altitude]

and $\triangle ABD = \frac{1}{2} AB.DQ$. [$AB$ is base and DQ is altitude]

$\therefore \frac{1}{2} AB.CP = \frac{1}{2} AB.DQ$ [$\because \triangle ABC = \triangle ABD$]

$\therefore CP = DQ$.

Again, CP and DQ are perpendiculars to the same straight line AB.

$\therefore CP \parallel DQ$.

$\therefore CPQD$ is a parallelogram.

$\therefore DC \parallel PQ$ or AB [opposite sides of a parallelogram.]

Alternative Proof :

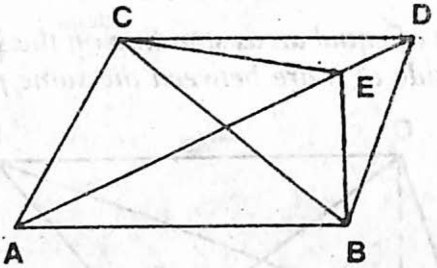


Fig. 51

Let the areas of $\triangle ABC$ and $\triangle ABD$ be equal and they lie on the same side of the common base AB. C, D are joined.

It is required to prove that $AB \parallel CD$.

Proof : Let CD be not parallel to AB. Then the straight line through the point C parallel to AB will intersect AD or AD produced. Let it intersect AD at the point E. B, E are joined.

$\because$ $\triangle ABC$ and $\triangle ABE$ are on the same base AB and between the same parallels AB and CE,

$\therefore \triangle ABC = \triangle ABE.$

But $\triangle ABC = \triangle ABD$ (by hypothesis)

$\therefore \triangle ABD = \triangle ABE$

But $\triangle ABE$ is a part of $\triangle ABD$ and hence the area of $\triangle ABE$ can not be equal to the area of $\triangle ABD$.

Hence the assumption that CD is not parallel to AB is not correct.

$\therefore AB \parallel CD.$

[*Remarks :* (1) This theorem is converse of Theorem 28. Hence it is proved in this way.

(2) Note that if triangles of equal areas have the same base and are on the same side of the base they will have same altitudes.]

Some applications :

1. BE and CF are two medians of $\triangle ABC$. *(i)* Prove that $\triangle BCE = \triangle BCF$; *(ii)* hence prove that $FE \parallel BC$.

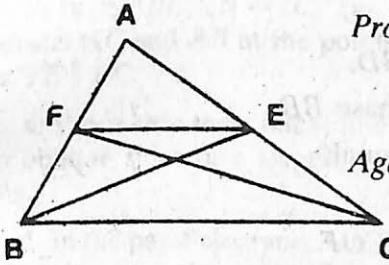


Fig. 52

Proof: $\because$ BE is a median

$\therefore \triangle BCE = \frac{1}{2} \triangle ABC$

Again, $\because$ CF is a median

$\therefore \triangle BCF = \frac{1}{2} \triangle ABC$

$\therefore \triangle BCE = \triangle BCF$ .... *(i)*

$\because$ the areas of $\triangle BCE$ and $\triangle BCF$ are equal and they are on the same side of the common base BC

$\therefore FE \parallel BC$. ....*(ii)*

[*Remark:* This is an alternative proof of Theorem 23. Since the proofs of the properties used in proving Theorem 29 are independent of Theorem 23, we can prove Theorem 23 in this way also.]

2. If a diagonal of a quadrilateral divides it into two triangles of equal area, it will bisect the other diagonal.

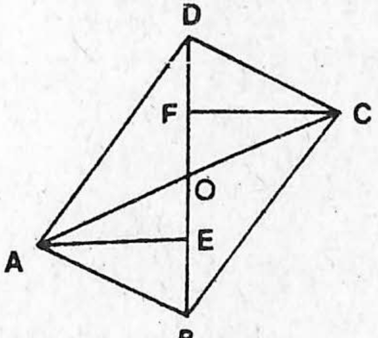


Fig. 53

Let the diagonal DB of the quadrilateral $ABCD$ divide it into two triangles of equal areas. That is, $\triangle ABD = \triangle CBD$. The two diagonals AC and BD intersect each other at the point O. It is required to prove that $AO = CO$.

Construction: Perpendiculars drawn from the points A and C respectively to BD intersect BD at the points E and F respectively.

Proof: $\triangle ABD = \triangle CBD$.

They are on the same base BD.

$\therefore$ their altitudes are equal.

$\therefore AE = CF$.

Now, in $\triangle AOE$ and $\triangle COF$

$\angle AOE = \angle COF$ (vertically opposite)

$\angle AEO = \angle CFO$ ($\because$ each is 1 right angle).

and $AE = CF$

$\therefore \triangle AOE \cong \triangle COF$.

$\therefore AO = CO$.

Exercise 11

1. Each diagonal of a quadrilateral divides it into two triangles of equal areas. Prove that the quadrilateral is a parallelogram.

2. In $\triangle ABC$, $AB = AC$. Perpendiculars from the points B and C to AC and AB interesect them at the points E and F respectively. Prove that $FE \parallel BC$.

3. In $\triangle ABC$, $\angle B = \angle C$. The bisectors of these two angles intersect AC and AB at the points E and F respectively. Prove that $FE \parallel BC$.

4. Prove that the straight line joining the mid-points of the two oblique sides of a trapezium is parallel to its two parallel sides.

5. In the parallelograms $ABCD$ and $AEFG$ of equal area $\angle A$ is common and the point E is on the side AB. Prove that $DE \parallel FC$.

6. $ABCD$ is a parallelogram and $ABCE$ is a quadrilateral. If the diagonal AC bisects the quadrilateral, prove that $AC \parallel DE$.

7. D is the mid-point of the side BC of $\triangle ABC$. P and Q are points on the sides BC and BA respectively such that $\triangle BPQ = \frac{1}{2} \triangle ABC$. Prove that $DQ \parallel PA$.

(2) Note that the points A, B and C are equidistant from the point O. Hence the circle drawn with O as centre and OA or OB or OC as radius will pass through the points A, B and C. This circle is called *circumcircle* of the triangle and the centre is called *circumcentre* of the triangle. Thus it may be said that the common point through which the three perpendicular bisectors of the three sides pass is the circumcentre of the triangle. The radius of that circle is called the *circumradius*.

(3) In fig. No. 54, the point O lies within the triangle as the triangle is acute. If the triangle was right-angled, O would have been the mid-point of the hypotenuse and if the triangle was obtuse, O would have been outside the triangle.

Theorem 31

Perpendiculars drawn from the vertices of a triangle to the opposite sides are concurrent.

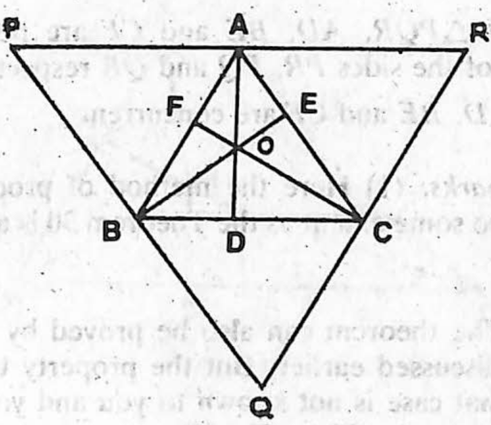


Fig. 55

Let *AD, BE* and *CF* be respectively the perpendiculars drawn from the three vertices *A, B* and *C* to the opposite sides *BC, CA* and *AB* of the triangle *ABC*.

It is required to prove that *AD, BE* and *CF* are concurrent.

Construction: Through the points *A, B* and *C* straight lines are drawn parallel to *BC, CA* and *AB* respectively to intersect each other at the points *P, Q* and *R* respectively.

Proof: By construction each of *APBC, ABCR* and *ABQC* is a parallelogram.

From the parallelograms *APBC* and *ABCR*, we get

$AP = BC$ and $AR = BC$

$\therefore AP = AR$

$\therefore$ *A* is the mid-point of *PR*.

Similarly, it can be shown that

B is the mid-point of *PQ*

and *C* is the mid-point of *QR*.

$\because$ BC PR and AD

$\therefore$ $AD \perp PR$

Similarly, $BE \perp PQ$ and $CF \perp QR$.

$\therefore$ in $\triangle PQR$, AD, BE and CF are perpendicular bisectors of the sides PR, PQ and QR respectively.

$\therefore$ AD, BE and CF are concurrent.

[*Remarks:* (1) Here the method of proof has been changed to some extent as the Theorem 30 is applicable in this case.

(2) The theorem can also be proved by the general method discussed earlier. But the property that is to be used in that case is not known to you and you will learn that property in Class X. Hence you can prove the theorem by that method in Class X.

(3) The point of concurrence is called *orthocentre*. In this case, O is the *orthocentre* of $\triangle ABC$.

(4) It has already been said earlier that the triangle obtained by joining the feet of the perpendiculars is called *pedal triangle*.

(5) Note that the perpendiculars bisect the angles of the pedal triangle.]

Theorem 32

Internal bisectors of the angles of a triangle are concurrent.

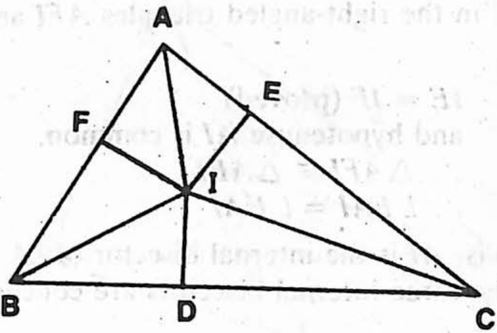


Fig. 56

Let ABC be a triangle.

It is required to prove that the internal bisectors of $\angle A$, $\angle B$ and $\angle C$ are concurrent.

Let the internal bisectors of $\angle B$ and $\angle C$ interset at the point I. A, I are joined. It will be proved that the three internal bisectors will be concurrent if it can be proved that AI is the bisector of $\angle A$.

Construction: ID, IE and IF are drawn perpendiculars from the point I to BC, CA and AB respectively.

Proof: In $\triangle BID$ and $\triangle BIF$

$\angle IBD = \angle IBF$ [$\because$ BI is the internal bisector of $\angle B$]

$\angle IDB = \angle IFB$ [$\because$ each is 1 right angle]
and IB is common.

$\therefore$ $\triangle BID \cong \triangle BIF$
$\therefore$ $ID = IF$

Similarly, from $\triangle CID$ and $\triangle CIE$ it can be shown that $ID = IE$.

$$\therefore\ IE = IF$$

Now, in the right-angled triangles AFI and AEI, we have

$IE = IF$ (proved)
and hypotenuse AI is common.
$\therefore\ \triangle AFI \cong \triangle AEI$
$\therefore\ \angle FAI = \angle EAI$

That is, AI is the internal bisector of $\angle A$.
$\therefore$ the three internal bisectors are concurrent.

[*Remark:* Note that $ID = IE = IF$; that is the circle with centre I and radius ID or IE or IF will pass through the points D, E and F. This circle is called the *incircle* of the triangle and the centre of this circle is called the *incentre*. The radius of the circle is called *inradius*. The point of concurrence of the internal bisectors of the angles of a triangle is the incentre.]

Theorem 33

The medians of a triangle are concurrent.

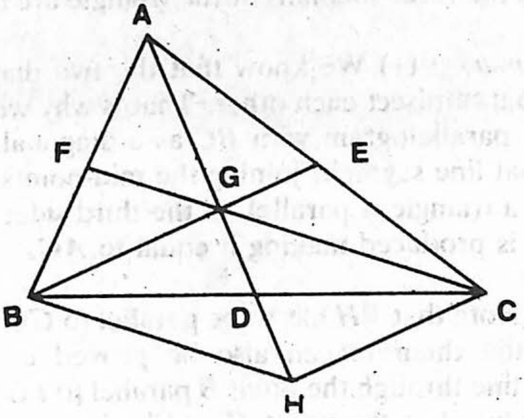


Fig. 57

Let in $\triangle ABC$, the medians BE and CE intersect at the point G.

It is required to prove that the three medians are concurrent.

Let A, G be joined and then produced, which interset BC at the point D. The concurrence of the three medians will be proved, if we prove that D is the mid-point of BC.

Construction: AD is produced to the point H in such a way that $GH = AG$.

B, H and C, H are joined.

Proof: In $\triangle ABH$, F and G are the mid-points of the sides AB and AH respectively.

$\therefore FG \parallel BH$ i.e., $GC \parallel BH$

Again, in $\triangle AHC$, G and E are the mid-points of the sides AH and AC respectively.

$\therefore GE \parallel HC$ i.e., $BG \parallel HC$

$\therefore BGCH$ is a parallelogram.

Since the diagonals of a parallelogram bisect each other

$\therefore$ D is the mid-point of BC.

$\therefore$ the three medians of the triangle are concurrent.

[*Remarks:* (1) We know that the two diagonals of a parallelogram bisect each other. That is why we have tried to get a parallelogram with BC as a diagonal. Again we know that line segment joining the mid-points of any two sides of a triangle is parallel to the third side; that is why the line is produced making it equal to AG.

(2) Note that BH has to be parallel to GC i.e., to FG. Hence the theorem can also be proved by drawing a straight line through the point B parallel to FG to intersect AD produced at the point H and by joining H, C. Try to prove the theorem in this way.

(3) The point of concurrence of the three medians is called the *centroid*. In Fig. 57, the point G is the centroid of $\triangle ABC$.]

Corollary: The medians are trisected at the centroid.

In Fig. 57, $GD = \frac{1}{3}AD$, $FG = \frac{1}{3}CF$, $EG = \frac{1}{3}BE$.

$\because$ the diagonals of the parallelogram $BGCH$ bisect each other

$\therefore$ D is the mid-point of GH.

$$\therefore GD = \tfrac{1}{2}GH = \tfrac{1}{2}AG.$$
$$\therefore 2\,GD = AG$$
$$3\,GD = AD$$
$$\therefore GD = \tfrac{1}{3}AD$$

[*Remark:* We can state this property in the following way also: 'A median is divided at the centroid in the ratio 2 : 1.']

Some applications :

1. The medians AD, BE and CF of $\triangle ABC$ intersect each other at the point G. Prove that

(i) $\triangle GBC = \frac{1}{3}\,\triangle ABC$,
(ii) $\triangle GBD = \frac{1}{6}\,\triangle ABC$.

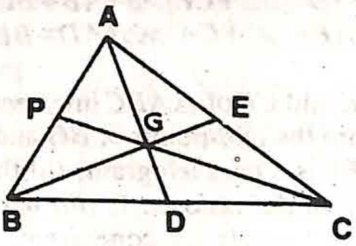


Fig. 58

Proof: AD is a median of $\triangle ABC$

$$\therefore\ \triangle ABD = \triangle ACD$$

Again, GD is a median of $\triangle GBC$.

$$\therefore\ \triangle GBD = \triangle GCD$$
$$\therefore\ \triangle ABD - \triangle GBD = \triangle ACD - \triangle GCD$$
$$\therefore\ \triangle AGB = \triangle AGC.$$

Similarly, it can be proved that $\triangle AGC = \triangle BGC$

$$\therefore\ \triangle BGC = \triangle AGC = \triangle AGB$$
$$= \tfrac{1}{3}(\triangle BGC + \triangle AGC + \triangle AGB)$$
$$= \tfrac{1}{3}\,\triangle ABC \qquad \text{.... } (i)$$

Again, $\triangle GBD = \frac{1}{2}\,\triangle BGC$

$$\therefore\ \triangle GBD = \tfrac{1}{2}(\tfrac{1}{3}\,\triangle ABC)$$
$$= \tfrac{1}{6}\,\triangle ABC \qquad \text{.... } (ii)$$

Exercise 12

1. The three medians AD, BE and CF of $\triangle ABC$ intersect each other at the point G. Area of $\triangle ABC$ is 36 sq. cm. *(i)* What is the area of $\triangle AGB$? *(ii)* What is the area of $\triangle CGE$? *(iii)* What is the area of the quadrilateral $BDGF$? Give reason in each case.

2. AD, BE and CF are the medians of $\triangle ABC$. Prove that

(i) $4\,(AD+BE+CF)>3\,(AB+BC+CA)$

(ii) $3\,(AB+BC+CA)>2\,(AD+BE+CF)$.

3. Medians BE and CF of $\triangle ABC$ intersect each other at the point G. P and Q are the mid-points of BG and CG respectively. Prove that *(i)* $PQEF$ is a parallelogram; *(ii)* the point G divides each of BE and CF in the ratio $2:1$; *(iii)* hence prove that the three medians of the triangle are concurrent.

2.8 Pythagoras' Theorem :

The property stated below is called Pythagoras' theorem. At present you are required to know the statement of the theorem and to verify it through practical methods.

Statement of Pythagoras' theorem:

In a right-angled triangle, the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the other two sides.

By means of a figure, the statement can be expressed in the following way.

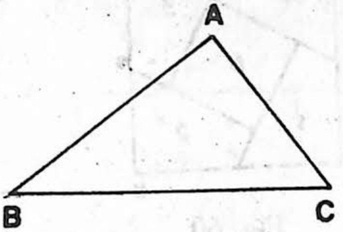


Fig. 59

In $\triangle ABC$, $\angle A$ is a right angle.

$\therefore$ the side BC is hypotenuse.

Then $BC^2 = AB^2 + AC^2$

[Remark: The area of the square on BC is written as BC^2.]

Verification by the method of dissection :

In Fig. 60, ABC is a right-angled triangle in which BC is the hypotenuse and $AC > AB$.

Square on BC is $BCKL$, square on AB is $ABDE$ and square on AC is $ACFG$.

The point of intersection of the diagonals of the square $ACFG$ is O. The straight line through the point O, parallel to BC intersects AG and CF at the points P and Q respectively. Again, the straight line through the point O perpendicular to PQ intersect AC and GF at the points R and S respectively. As a result, the square $ACFG$ is divided into four parts which are marked 1, 2, 3, 4. The square $ABDE$ is marked 5.

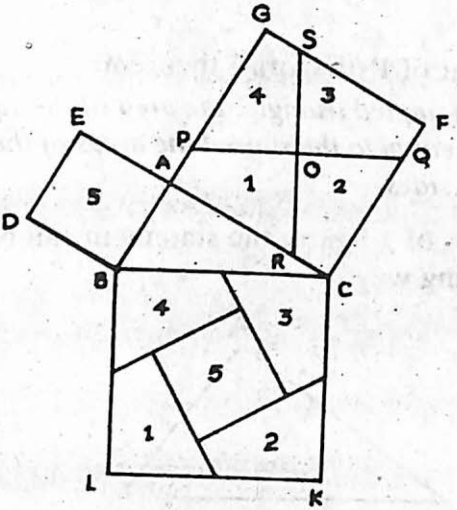


Fig. 60

On a thick paper draw a figure like Fig. 60 and cut it out. Now cut out the squares from this figure. Cut the square $ACFG$ along the lines PQ and RS dividing it in four parts. Now place the parts 1, 2, 3, 4 and 5 properly on the square $BCKL$. Note that those five parts together exactly fit in the square. Thus we find that

$$BC^2 = AB^2 + AC^2$$

You can also verify this property in the following way. Let $BC = a$, $CA = b$ and $AB = c$.

Draw a square $PQRS$ whose side is $(b+c)$. Take points E, F, G, H on sides PQ, QR, RS and SP respectively such that $PE = QF = RG = SH = b$.

Then you willl get four right-angled triangles having a as hypotenuse and b and c as other two sides. The remaining part of the figure is a^2.

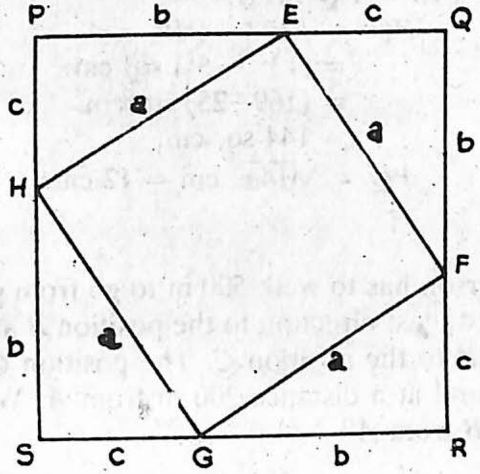


Fig. 61

$$\therefore (b + c)^2 = a^2 + 4.\tfrac{1}{2}bc$$
$$b^2 + c^2 + 2bc = a^2 + 2bc$$
$$b^2 + c^2 = a^2$$

[*Remark:* Hindu mathematician Bhaskaracharyya has contribution in the verification of Pythagoras' theorem by the method of dissection.]

Some application :

1. BC is the hypotenuse of the right-angled triangle ABC. If $AB = 3$ cm and $AC = 4$ cm, what is the length of BC?

According to Pythagoras' theorem

$$\begin{aligned} BC^2 &= AB^2 + AC^2 \\ &= (3^2 + 4^2) \text{ sq.cm.} \\ &= (9 + 16) \text{ sq. cm.} \\ &= 25 \text{ sq. cm.} \\ \therefore BC &= \sqrt{25} \text{ cm} = 5 \text{ cm.} \end{aligned}$$

2. PR is the hypotenuse of the right-angled triangle PQR. If $PR = 13$ cm, $QR = 5$ cm, $PQ =$ how many cm?

According to Pythagoras theorem

$$PR^2 = PQ^2 + QR^2$$

$$\therefore PQ^2 = PR^2 - QR^2$$

$$= (13^2 - 5^2) \text{ sq. cm.}$$

$$= (169-25) \text{ sq. cm.}$$

$$= 144 \text{ sq. cm.}$$

$$\therefore PQ = \sqrt{144} \text{ cm} = 12 \text{ cm.}$$

3. A person has to walk 500 m to go from position A in the north of east direction to the position B and then to the west of B to the position C. The position C is to the north of A and at a distance 200 m from A. What is the distance of B from A?

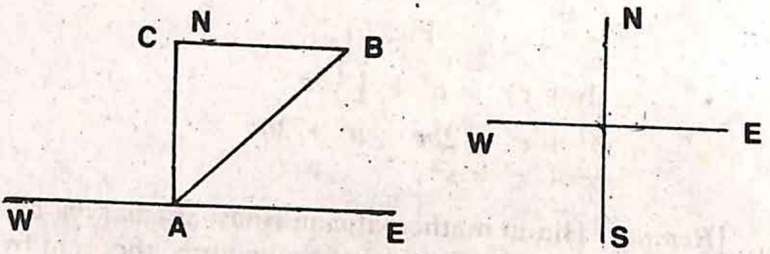


Fig. 62

Let $AB = x$ metres

$\therefore BC = (500 - x)$ metres

$\because \triangle ABC$ is right-angled where $\angle C$ is a right angle.

$\therefore AB^2 = BC^2 + AC^2$

or, $x^2 = (500-x)^2 + 200^2$

or, $x^2 = (250000 - 1000x + x^2 + 40000$

or, $1000x = 290000$

or, $x = 290$

$\therefore$ distance of B from $A = 290$ metres.

Exercise 13

1. The lengths of the two sides adjacent to the right angle of a right-angled triangle are 60 cm and 11 cm; what is the length of the hypotenuse?

2. The lengths of the hypotenuse and another side of a right-angled triangle are 17 cm and 8 cm respectively; what is the length of the third side?

3. In a right-angled triangle, the length of one side is 35 cm, the difference of the lengths of the hypotenuse and the third side is 25 cm. Find the lengths of the hypotenuse and the third side.

4. From the same place one man goes 56 m towards north-east and another man goes 33 m towards south-east. What is the distance between their final positions?

5. *B* is to the west and *C* is to the south of *A*. A man has to walk 176 metres to go from *A* to *B*, *B* to *C* and *C* to *A*. But he has to walk 121 metres to go from *A* to *C* via *B*. Find the distances from *A* to *B*, *B* to *C* and *C* to *A*.

CHAPTER 3

Problems

Construction of figures equal in area :

Problem 3

To construct a parallelogram equal in area to a given triangle with one of its angles equal to a given angle.

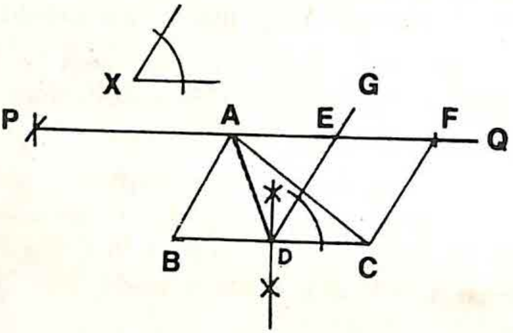


Fig. 63

Let ABC be a given triangle and X be a given angle.

We have to construct a parallelogram equal in area to $\triangle ABC$ with one of its angles equal to $\angle X$.

Method of construction: The side BC of $\triangle ABC$ is bisected at the point D. Through the point A a straight line PQ is drawn parallel to BC. $\angle CDG$ is constructed equal to $\angle X$, in such a way that the points G and A are on the same side of BC. Let DG intersect PQ at the point E. From EQ the portion EF is cut off making it equal to DC. C, F are joined.

Then $CDEF$ is required parallelogram.

Proof: A, D are joined

$$\because DC \parallel EF \text{ and } DC = EF$$
$$\therefore CDEF \text{ is a parallelogram}$$

$\angle CDE$ of the parallelogram $CEDF$ *is equal to the given* $\angle X$.

Again, $\triangle ADC$ and parallelogram $CDEF$ are on the same base DC and between the same parallels DC and AQ.

$\triangle ADC = \frac{1}{2} \square CDEF$

But AD is a median of $\triangle ABC$

$$\therefore \triangle ADC = \tfrac{1}{2} \triangle ABC$$

$$\therefore \square CDEF = \triangle ABC$$

That is, the area of the parallelogram $CDEF$ is equal to the area of $\triangle ABC$ and one of its angles, $\angle CDE$ is equal to $\angle X$.

[*Remarks:* (1) Note that of the three methods of constructing parallel lines that you have learnt in Class VIII one has been applied here. Actually this method is based on two properties: 'if two pairs of opposite sides of a quandrilateral are parallel, it is a parallelogram' and 'opposite sides of a parallelogram are parallel'; You could draw the parallel line by any other method.

parallelogram are parallel'. You could draw the parallel line by any other method.

(2) In the parallelogram $CDEF$ we have cut off EF equal to DC. Here we have applied the property: 'if a pair of opposite sides of a quadrilateral are parallel and equal, the quadrilateral is a parallelogram'. We could also use the property: 'two pairs of opposite sides of a parallelogram are parallel'; i.e., we could construct the straight line through the point C parallel to DE by a different method.

(3) In this construction, we need to know the position of D, the midpoint of BC. For that instead of constructing the perpendicular bisector we could proceed in the following manner.

With centres B and C and radii equal to AC and AB respectively we draw two arcs which intersect each other at the point P. A, P are joined to intersect BC at the point D. Then D is the mid-point of BC.

By construction, $ABPC$ is a parallelogram. Hence its diagonals AP and BC bisect each other.

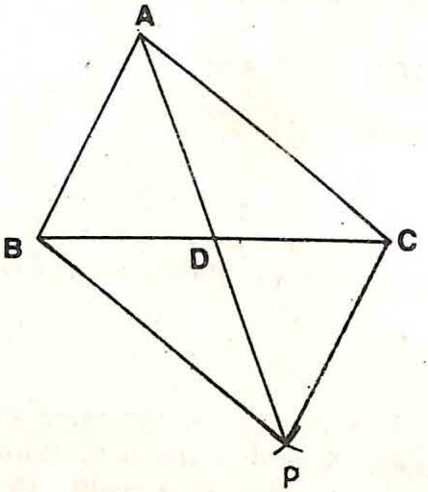


Fig. 64

(4) If the proof is not required there is no need for joining A, D in Fig. 63.

Some Applications :

Construct a rectangle equal in area to a given triangle.

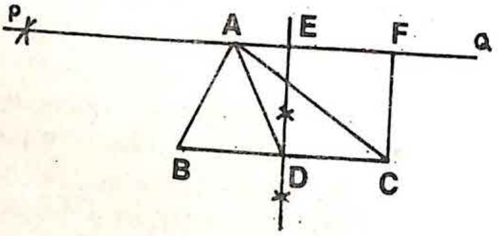


Fig. 65

Let ABC be a given triangle.

It is required to construct a rectangle equal in area to $\triangle ABC$.

Method of construction: We construct the perpendicular bisector of the side BC of $\triangle ABC$ to intersect BC at the point D. Then we construct the line PQ through the point A parallel to BC. The perpendicular bisector intersects PQ at the point E. From EQ the portion EF is cut off equal to DC. C, F are joined.

Then $CDEF$ is the required rectangle.
Proof: $\because$ $DC = EF$ and $DC \parallel EF$

$\because$ $CDEF$ is a parallelogram.

Again, $\angle CDE = 1$ rt. angle ($\because$ $DE \perp BC$)

$\therefore$ CDEF is a rectangle

A, D are joined

$\triangle ADC = \frac{1}{2} \triangle ABC$ ($\because$ AD is a median of $\triangle ABC$)

Again, $\triangle ADC$ and rectangle $CDEF$ are on the same base DC and between the same parallels DC and EF,

$\therefore$ $\triangle ADC = \frac{1}{2} \square\, CDEF$

$\therefore$ $\square\, CDEF = \triangle ABC$

[*Remark :* Since angles of a rectangle are right angles, therefore, there is no need of mentioning any angle in this case.]

Exercise 14

(Write the method of construction and give the proof 1-7.)

1. Construct a triangle with sides 5 cm, 8 cm and 11 cm. Construct a parallelogram equal in area to the above triangle and with one of its angles equal to 45°

2. Construct $\triangle ABC$, where $AB = 6$ cm, $BC = 9$ cm and $\angle ABC = 60°$. Construct a parallelogram equal in area to $\triangle ABC$ with one of its angles equal to 60° and with one of its sides equal to half of AC.

3. Consturct a triangle PQR where $\angle PQR = 30°$, $\angle PRQ = 75°$ and $QR = 7.5$ cm. Construct a rectangle equal in area to $\triangle PQR$.

4. Construct an equilateral triangle with side 6·5 cm. Construct a parallelogram equal in area to it with one of its angles equal to 55°.

5. Construct an isosceles triangle with each of its equal sides as 8 cm and with its base 5 cm. Construct a parallelogram equal in area to that triangle with one of its angles equal to one of the equal angles of the triangle and with one of its sides equal to half of one of the equal sides of the triangle.

6. Construct an isosceles triangle where the length of one of its equal sides is 9·2 cm and one of the equal angles is 60°. Construct a rectangle equal in area to it with one of its sides equal to half the base of the triangle.

7. Construct an equilateral triangle of side 7·6 cm. Construct a rectangle equal in area to that triangle.

8. In $\triangle ABC$, $AB = 6$ cm, $AC = 8$ cm and $BC = 10$ cm. Construct a striaght line through the point A parallel to BC. Bisect BC at the point D. Construct a line segment DE of length 9·5 cm such that the point E lies above parallel straight line through the point A. Cut the portion EF from that parallel line equal to DC, such that the points F and C are on the same side of DE. Join C, F. What kind of quadrilateral is $CDEF$? What is the relationship of this quadrilateral with $\triangle ABC$? Give reasons.

Problem 4

To construct a triangle equal in area to a given quadrilateral.

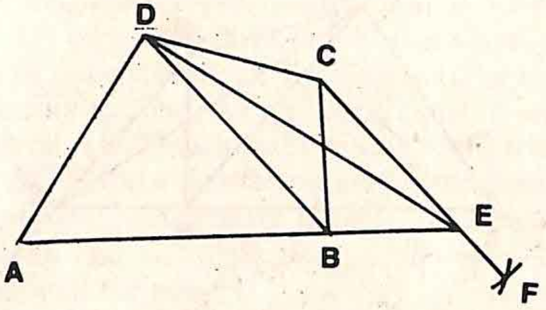


Fig. 66

Let $ABCD$ be the given quadrilateral.

It is required to construct a triangle equal in area to the quadrilateral $ABCD$.

Method of construction: The diagonal DB is drawn. Through the point C, CF is drawn parallel to DB. CF intersects AB produced at the point E. D, E are joined. Then $\triangle ADE$ is the required triangle.

Proof: $\triangle DEB$ and $\triangle DCB$ are on the same base DB and between the same parallels DB and CF.

$$\therefore\ \triangle DEB = \triangle DCB$$

$$\therefore\ \triangle ADB + \triangle DEB = \triangle ADB + \triangle DCB$$

$$\triangle ADE = \text{quadrilateral } ABCD$$

[*Remarks:* (1) To construct the parallel line an arc is drawn with centre B and radius DC and another arc is drawn with centre C and radius DB. The parallel line could also be drawn by a different method.

(2) Let us now discuss why we proceeded to construct the triangle by the above method.

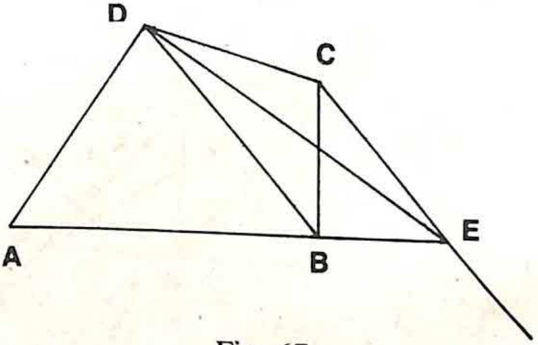


Fig. 67

If two triangles of equal area $\triangle DEB$ and $\triangle DCB$ on the side DB are on the same side of DB, we know that DB will be parallel to CE. Now if EB is produced to any point A and A, D are joined, ADE will be a triangle and $ABCD$ will be a quadrilateral and they will have equal areas. Here note that DB is diagonal of quadrilateral $ABCD$ and CE is parallel to that diagonal DB. E is on AB produced. From this it can be understood why we followed the above method for the construction.

(3) Note that the side AB of the quadrilateral and the side AE of the triangle are on the same straight line and D

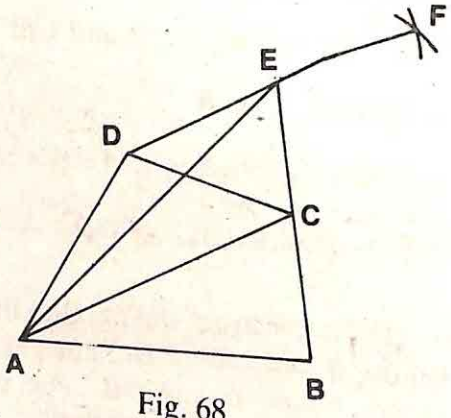


Fig. 68

is vertex of both. Thus we can construct the triangle equal in area to the quadrilateral in such a way that one side of the quadrilateral and one side of the triangle will be on the same straight line and any one of the other two vertices of the quadrilateral will be the vertex, of the triangle. In that case the diagonal of the quadrilateral is to be drawn through that vertex and through the other vertex a straight line is to be drawn parallel to the diagonal. For example, if it is necessary to construct a triangle equal in area to the quadrilateral *ABCD* such that one side of the triangle and the side *BC* of the quadrilateral are on the same straight line and *A* is the vertex of the triangle, the diagonal *AC* is to be drawn and a straight line parallel to *AC* is to be drawn through the point *D*.

(4) By applying Problems 4 and 3 it is possible to construct a parallelogram or a rectangle equal in area to a given quadrilateral.

(5) Note that the number of sides of a quadrilateral is four and that of a triangle is three. Thus we find that in construction 4 from a given polygon another polygon is constructed whose area is equal to that of the given polygon and whose number of sides is one less than that of the given polygon. Hence by this method it is possible to construct a quadrilateral equal in area to a pentagon, a pentagon equal in area to a hexagon, etc. Applying this method repeatedly it will be possible to construct a triangle equal in area to any given polygon.]

Some applications :

1. Construct a parallelogram equal in area to a given quadrilateral with one of its angles equal to a given angle.

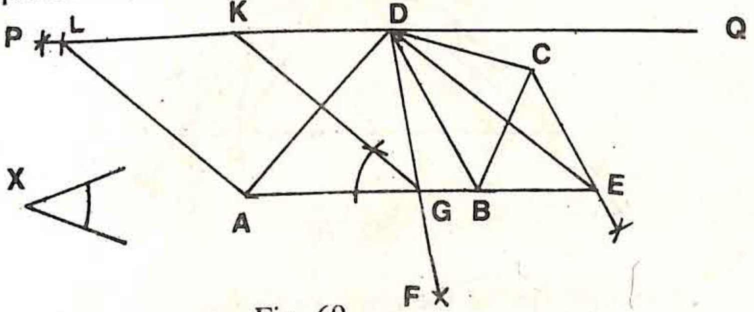


Fig. 69

Let $ABCD$ be the given quadrilateral and $\angle X$ be the given angle.

It is required to construct a parallelogram equal in area to the quadrilateral $ABCD$ with one of its angles equal to $\angle X$.

Method of Construction: Diagonal DB is drawn. Through the point C a straight line is drawn parallel to DB to intersect AB produced at the point E. D, E are joined.

Then $\triangle ADE$ = quadrilateral $ABCD$

Through the point D the straight line PQ is drawn parallel to AB. G, the mid-point of AE, is marked. $\angle AGK$ is constructed equal to $\angle X$ where GK intersects PQ at the point K. From PQ the portion KL is cut off making it equal to GA and in such a way that the points A and L are on the same side of GK.

Then $AGKL$ is the required parallelogram.

Proof: By construction $\triangle ADE$ = quadrilateral $ABCD$.

Again, by construction parallllelogram $AGKL$ = $\triangle ADE$.

$\therefore$ Parallelogram $AGKL$ = quadrilateral $ABCD$.

2 Construct a triangle equal in area to a given pentagon.

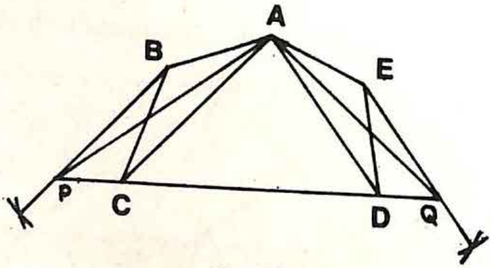


Fig. 70

Let $ABCDE$ be the given pentagon.

It is required to construct a triangle equal in area to the pentagon $ABCDE$.

Method of construction:

Diagonals AC and AD are drawn. Through the points B and E lines are drawn parallel to AC and AD respectively. DC produced intersects the first straight line at the point P and CD produced intersects the second straight line at the point Q. A, P and A, Q are joined.

Then $\triangle APQ$ is the required triangle.

Proof: $\triangle APC$ and $\triangle ABC$ are on the same base AC and between the same parallels AC and BP.

$$\therefore\ \triangle APC = \triangle ABC$$

Again, $\triangle AQD$ and $\triangle AED$ are on the same base AD and between the same parallels AD and EQ.

$$\therefore\ \triangle AQD = \triangle AED$$

$$\therefore\ \triangle APC + \triangle ACD + \triangle AQD = \triangle ABC + \triangle ACD + \triangle AED$$

$$\therefore\ \triangle APQ = \text{pentagon } ABCDE.$$

Exercise 15

(In each of the following cases write the method of construction and the proof.)

1. Construct a quadrilateral $ABCD$ with $AB = 3$ cm, $BC = 4$ cm, $CD = 4\cdot 2$ cm, $DA = 3\cdot 8$ cm and diagonal $AC = 6$ cm. Construct a triangle equal in area to this quadrilateral.

2. Construct a quadrilateral $ABCD$ with $BC = 6$ cm, $AB = 4$ cm, $CD = 3$ cm, $\angle ABC = 60°$, $\angle BCD = 45°$. Construct a triangle, equal in area to the above quadrilateral, with AB as one of its sides and another side lying along BC.

3. Construct a quadrilateral $ABCD$ with $BC = 5\cdot 3$ cm, $CD = 4$ cm, diagonal $BD = 10$ cm, $\angle ADB = 45°$ and $\angle ABC = 75°$ Construct a rectangle equal in area to the above quadrilateral.

4. Construct a rectangle $ABCD$ with $AB = 5$ cm and $BC = 3$ cm and construct a triangle equal in area to this rectangle.

5. Draw a pentagon $ABCDE$ and construct a triangle equal in area to this pentagon with one of the vertices of the triangle at the point C.

6. Draw a square with 5 cm as its side. Draw a triangle, equal in area to that square.

7. AD and BC are perpendiculars on side AB of a quadrilateral $ABCD$ and $AB = 5$ cm, $AD = 7$ cm and $BC = 4$ cm. Draw the quadrilateral. Draw a triangle equal in area to that quadrilateral and whose one angle is 30°.

Miscellaneous Exercise 1

In some of the following questions possible answers are given within brackets. Write the correct answers from these and give reasons in support of your answer.

1. $ABCD$ is a parallelogram with $AB = 5$ cm, $BC = 3$ cm and $\angle ABC = 105°$.

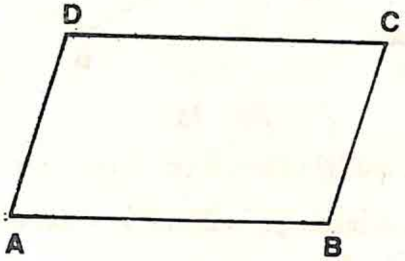


Fig. 71

CD = how much? (5 cm, 3 cm, 2·5 cm, 1·5 cm)
AD = how much? (5 cm, 3 cm, 2·5 cm, 1·5 cm)
$\angle BAD$ = how much? (105°, 75°, 52·5°, 37·5°)
$\angle BCD$ = how much? (105°, 75°, 52·5°, 37·5°)
$\angle CDA$ = how much? (105°, 75°, 52·5°, 37·5°)

2. $ABCD$ is a parallelogram with $AD = 7$ cm, $AB = 4$ cm and $PO = 3$ cm.

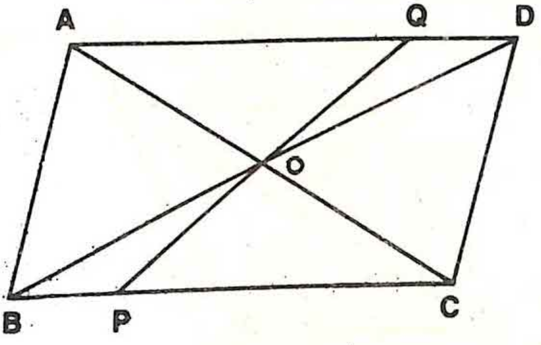


Fig. 72

PQ = how much? (7 cm, 6 cm, 4 cm, 3 cm)

3. In the quadrilateral $ABCD$, $AO = OC$, $BO = OD$; $AB =$ 5 cm, $AD = 3$ cm.

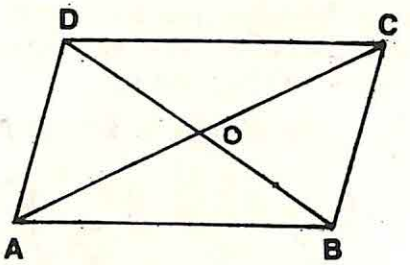


Fig. 73

$DC =$ how much? (5 cm, 4 cm, 3 cm, 2·5 cm)

4. $ABCD$ is a rectangle with $AB = 4$ cm, $OD = 2\cdot5$ cm.

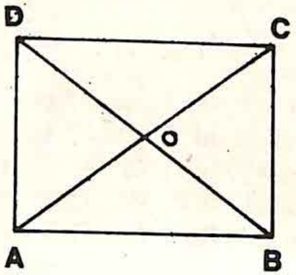


Fig. 74

$BC =$ how much? (4 cm, 2·5 cm, 5 cm, 3 cm)

5. In the rhombus $ABCD$, $\angle BAD = 60°$, $BC = 4$ cm.

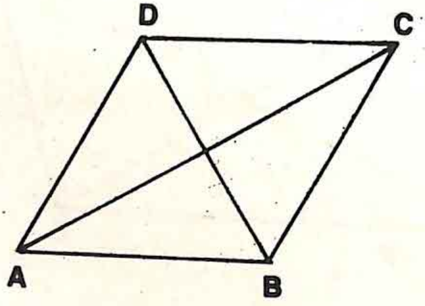


Fig. 75

$BD =$ how much? (2 cm, 4 cm, 5 cm, 6 cm)

6. *ABCD* is a square.
$\angle BAC$ = how much? (30°, 45°, 60°)

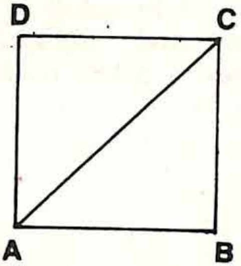


Fig. 76

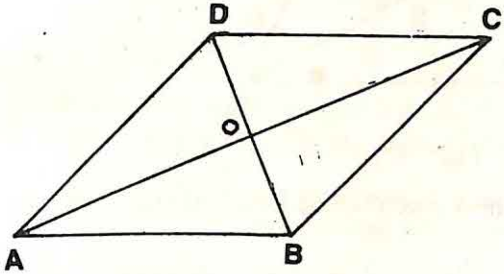


Fig. 77

7. *ABCD* is a rhombus.
$AC = 24$ cm. $BD = 10$ cm.
AB = how much?
(5 cm, 12 cm, 13 cm, 26 cm)

8. In $\triangle ABC$, P and Q are the mid-points of AB and AC respectively, $AB = 2$ cm, $BC = 3$ cm and $AC = 2{\cdot}6$ cm.

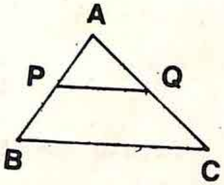


Fig. 78

PQ = how much? (1 cm, 1·5 cm, 1·3 cm)

9. In $\triangle ABC$, P is the mid-point of AB, $PQ \parallel BC$, $AP = 2$ cm, $AQ = 1{\cdot}5$ cm, $PQ = 1{\cdot}7$ cm.
AC = how much?
(4 cm, 3 cm, 3·4 cm)

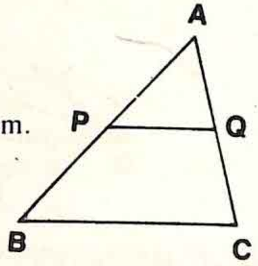


Fig. 79

10. In $\triangle ABC$. P and Q are the mid-points of AB and AC respectively. BQ and CP intersect each other at the point R. Area of $\triangle ABC$ is 24 sq.cm.

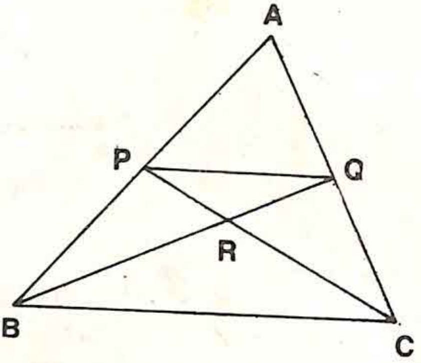


Fig. 80

Area of $\triangle PBQ$ = how much? (4 sq. cm, 6 sq.cm, 12 sq.cm, 18 sq.cm)

11. $ABDE$ is a parallelogram. F is the mid-point of ED. Area of $\triangle ABC$ is 20 sq.cm.

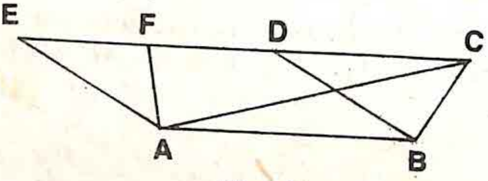


Fig. 81

Area of $\triangle AEF$ = how much? (20 sq.cm, 10 sq.cm, 5 sq.cm, 4 sq.cm)

12. In $\triangle ABC$, $AB = 8$ cm, $AC = 6$ cm and $BC = 10$ cm. Bisectors of $\angle ABC$ and $\angle BCA$ intersect each other at the point I.

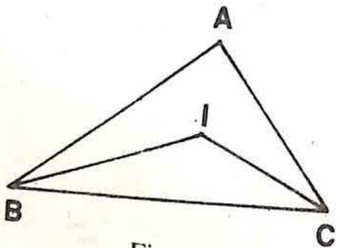


Fig. 82

ID, IE, IF are perpendiculars to BC, CA and AB respectively. $IF = 2$ cm.

ID = how much? (2 cm, 3 cm, 4 cm, 5 cm)
IE = how much? (2 cm, 3 cm, 4 cm, 5 cm)

13. In $\triangle ABC$, $AB = 4$ cm, $BC = 5$ cm, $CM = 3$ cm. Perpendicular bisectors of AB and BC intersect each other at the point O.

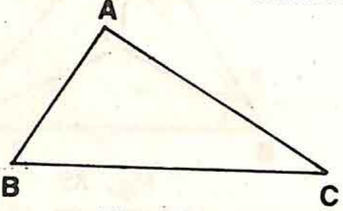


Fig. 83

BO = how much? (1 cm, 1·5 cm, 2 cm, 2·5 cm)
AO = how much? (1 cm, 2 cm, 2·5 cm, 1·5 cm)

14. In $\triangle ABC$, the mediams AD and BE intersect at the point G. CG produced intersects AB at the point F. $AB = CF =$ 6 cm. Area of $\triangle ABC$ is 9 sq.cm.

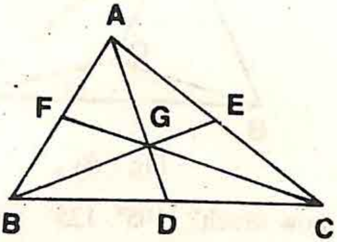


Fig. 84

AF = how much? (3 cm, 2 cm, 1·5 cm, 1 cm)
GF = how much? (3 cm, 2 cm, 1·5 cm, 1 cm)

Area of $\triangle GBC$ = how much?
(6 sq.cm, 4·5 sq.cm, 3 sq.cm, 1·5 sq.cm)

Area of $\triangle GEA$ = how much?
(6 sq.cm, 4·5 sq.cm, 3 sq.cm, 1·5 sq.cm)

Area of quadrilateral $BDGF$ = how much?
(6 sq.cm, 4·5 sq.cm, 3 sq.cm, 1·5 sq.cm)

15. In $\triangle ABC$, bisectors of $\angle ABC$ and $\angle BCA$ intersect each other at the point I.

$\angle ABC = 60°, \angle BCA = 50°$

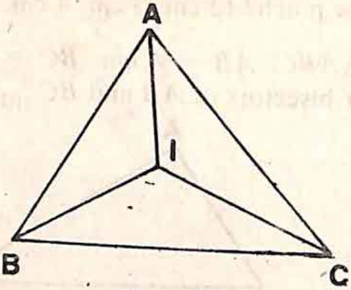


Fig. 85

$\angle BIC$ = how much? (160°, 150°, 140°, 125°)

16. In $\triangle ABC$, $\angle ABC = 65°$, $\angle BCA = 40°$, point of intersection of the perpendicular bisectors of the sides is O.

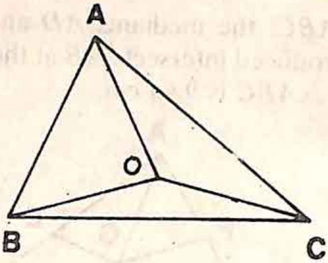


Fig. 86

$\angle BOC$ = how much? (105°, 125°, 150°, 165°)

17. In $\triangle ABC$, $\angle ABC = 70°$, $\angle BAC = 75°$; AD $\perp$ BC, BE $\perp$AC and $CF \perp$ AB.

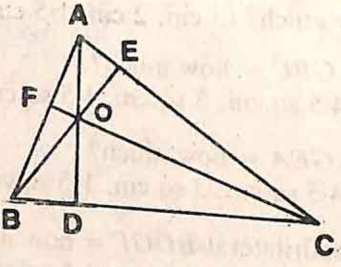


Fig. 87

$\angle BOA$ = how much? (105°, 110°, 125°, 145°)

18. In $\triangle ABC$, D, E and F are the mid-points of BC, CA and AB respectively, area of $\triangle ABC$ is 12 sq.cm.

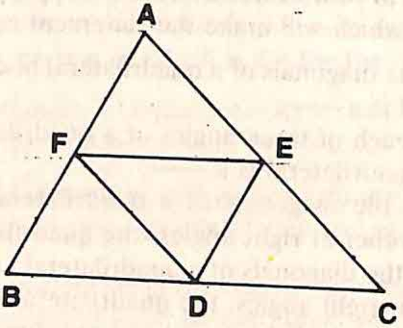


Fig. 88

Area of $\triangle DEF$ = how much? (3 sq.cm, 4 sq.cm, 6 sq.cm, 8 sq.cm)

Miscellaneous Exercise 2

1. Write in your exercise book, the appropriate words in the blank space which will make the statement correct :

(i) If the diagonals of a quadrilateral bisect each other, the quadrilateral is a ——.

(ii) If each of three angles of a quadrilateral be one right angle, the quadrilateral is a ——.

(iii) If the diagonals of a quadrilateral be unequal and bisect each other at right angles, the quadrilateral is a ——.

(iv) If the diagonals of a quadrilateral are equal and bisect each other at right angles, the quadrilateral is a ——.

(v) If two parallelograms be on the same base between the same parallels, they are —— in area.

(vi) If a parallelogram and a triangle are on the same base and have the same altitude, the area of the parallelogram is —— the area of the triangle.

(vii) The line segment joining the mid-points of any two sides of a triangle is —— to the third side and its length is —— the length of the third side.

(viii) A straight line drawn through the mid-point of one side of a triangle parallel to the second side in —— the third side.

2. To fill in the blanks choose appropriate word from the four words—circumcentre, incentre, orthocentre and centroid and write in your exercise book.

(i) In a right-angled triangle the mid-point of the hypotenuse is the —— of the triangle.

(ii) In a right-angled triangle the vertex at the right angle is the —— of the triangle.

(iii) If a point is taken on a median of a triangle is such a way that the length of the portion of the median from the vertex up to that point is twice the length of the other portion, that point is the —— of the triangle.

3. Put a tick (√) mark by writing the numbers of the correct statements in your exercise book and if the statements are wrong write their correct forms.

(i) Rhombus is a parallelogram.

(ii) Square is a particular form of rhombus.

(iii) If the diagonals of a quadrilateral are perpendicular to each other the quadrilateral is a rhombus.

(iv) If the four sides of a quadrilateral are equal, the quadrilateral is a square.

(v) The straight line joining the mid-point of a median of a triangle to any other vertex divides the side opposite to this vertex into two parts one of which is double the other.

(vi) If two triangles of equal areas have same base, the line joining their vertices will be parallel to the base.

(vii) If a parallelogram and a rectangle are on the same base and between the same parallels, they are equal in area.

4. Match the statements in the two columns to form correct sentences (each statements can be used once only).

square
rhombus
parallelogram
rectangle

is a particular form of quadrilateral
is a particular form of parallelogram
is a particular form of parallelogram
is a particular form of rectangle

5. Names of some four-sided figures are given in the left hand column and in the right hand column there is a list of four-sided figures obtained by joining the mid-points of the figures in the left hand column. Arrange the right hadn column correctly to match the left hand column.

(1) quadrilateral
(2) parallelogram
(3) rectangle
(4) square
(5) rhombus

(1) square
(2) rhombus
(3) rectangle
(4) parallelogram
(5) parallelogram

Miscellaneous Exercise 3

1. In the parallelogram $ABCD$, bisector of $\angle BAD$ also bisects $\angle DCB$. Prove that $ABCD$ is a rhombus.

2. In the parallelogram $ABCD$, the sum of $\angle BAD$ and $\angle DCB$ is two right angles. Prove that $ABCD$ is a rectangle.

3. Prove that the diagonal QS of the rhombus $PQRS$ bisects $\angle PSR$ and $\angle RQP$.

4. The diagonals of a rectangle are perpendicular to each other. Prove that the rectangle is a square.

5. In the parallelogram $ABCD$, the diagonal BD is perpendicular to AB. P and Q are the mid-points of AD and BC respectively. Prove that $BPDQ$ is a rhombus.

6. In the parallelogram $ABCD$, $AB>AD$. P is a point on AB, such that $AP = AD$. Prove that DP bisects $\angle ADC$. Find a point Q on AB produced such that QD bisects $\angle AQC$.

7. Parallelogram, $ABCD$ and $ABEF$ are on the same base and between the same parallels. P and Q are the points of intersection of the diagonals of the parallelograms $ABCD$ and $ABEF$ respectively. Prove that $PQ \parallel AB$.

8. $\triangle ABC$ and $\triangle ABD$ are equal in area and they are on the same side of the common base AB. Centroid of $\triangle ABC$ is G and centroid of $\triangle ABD$ is H. Prove that $GH \parallel AB$.

9. $\triangle ABC$ and $\triangle ABD$ are equal in area and are on the opposite sides of the common base AB. Prove that AB bisects CD.

10. Rectangle $ABCD$ and parallelogram $ABEF$ are equal in area and are on the opposite sides of the common base AB. The perpendicular drawn from the point F to BA intersects BA produced at the point G. Prove that $ADGF$ is a parallelogram.

11. Prove that of all the triangles having same base and equal areas, the isosceles triangle has the least perimeter.

12. O is any point inside $\triangle ABC$. Parallelograms $OAPB$, $OBQC$ and $OCRA$ are drawn with AB, BC and CA as diagonals respectively. Prove that $\triangle ABC \cong \triangle QPR$.

13. $ABCD$ is an isosceles trapezium with AD and BC as oblique sides. Prove that perpendicular bisector of DC is also the perpendicular bisector of AB. The perpendicular bisector of AD intersects the previous perpendicular bisector at the point O. Prove that $OA = OB = OC = OD$.

14. Each of different plane figures has some particular properties. The definition of each plane figure has been based on one of these properties. If the definitions were given on the basis of some other property it would have been necessary to prove the property on which the previous definition was based.

We assume below some such definitions.

(a) The quadrilateral, whose diagonals bisect each other, is a parallelogram.

(b) The quadrilateral, whose diagonals are equal in length and bisect each other, is a rectangle.

(c) The quadrilateral, whose diagonals bisect each other at right angles, is a rhombus.

(d) The quadrilateral, whose diagonals are equal in length and bisect each other at right angles, is a square.

Assuming the above definitions prove the following properties:

(i) Opposite sides of a parallelogram are parallel.

(ii) In a rectangle the opposite sides are parallel and each angle is a right angle.

(iii) All sides of a rhombus are equal.

(iv) In a square opposite sides are parallel, all sides are all equal and each angle is a right angle.

15. In proving Theorem 26, congruency of $\triangle BEC$ and AFD (fig. 34) was proved in the following way:

$\angle BEC$ = corresponding $\angle AFD$ ($\because$ $BE \parallel AF$)

$\angle BCE$ = corresponding $\angle ADF$ ($\because$ $AD \parallel BC$)

BE = AF (opposite sides of a parallelogram)
$\therefore \triangle BEC \cong \triangle AFD$.

If instead of proving in the above manner a student would have proceeded in the following way for the proof would it have been logical?

$BC = AD$ (opposite sides of a parallelogram)
$BE = AF$ (opposite sides of a parallelogram)

$\angle BEC =$ corresponding $\angle AFD$ ($\because$ BE ∥ AF)
$\therefore \triangle BEC \cong \triangle AFD$.

Give reasons in support of your answer.

If in your opinion the proof is not valid, can you prove the congruency of the triangles by showing that the corresponding sides of the two triangles are equal? Try if you can.

Miscellaneous Exercise 4

1. In which of the following cases will the quadrilateral $ABCD$ be a parallelogram according to the conditions given below?

(i) $AB = DC$ and $AB \parallel DC$.
(ii) $AB = DC$ and $AD = BC$.
(iii) $AB = DC$ and $\angle ADC = \angle ABC$.
(iv) $AB \parallel DC$ and $\angle ADC = \angle ABC$.
(v) $\angle ADC = \angle ABC$ and $\angle DAB = \angle DCB$.
(vi) $AB \parallel DC$ and $AD \parallel BC$.
(vii) AC and BD bisect each other.
(viii) $\triangle ADC \cong \triangle ABC$ or $\triangle BAD \cong \triangle BCD$.
(ix) $\triangle ADC \cong \triangle ABC$ and $\triangle BAD \cong \triangle BCD$.
(x) $\triangle ADC = \triangle BDC$ and $AD = BC$ (Two triangles lie on the same side of DC).
(xi) $\triangle ADC = \triangle BDC$ and $AB = DC$ (Two triangles lie on the same side of DC).

2. What form will the quadrilateral $ABCD$ be in each of the following cases if:

(i) AC and BD are unequal and bisect each other.
(ii) AC bisects BD at right angles .
(iii) AC and BD are unequal and bisect each other at right angles.
(iv) AC and BD are equal and bisect each other.
(v) AC and BD are equal and bisect each other at right angles.
(vi) AC and BD are equal and intersect each other at the point O, so that $AO = BO$, $CO = DO$.

3. In $\triangle ABC$, $\angle A$ is a right angle and $\angle C=30°$; if $BC=8$ cm, what is the length of the line segment AB?

4. Answer the following questions :

(i) In $\triangle ABC$, D and E are the mid-points of sides AB and AC respectively. What is the relation between the *straight line* DE and the straight line through the point D parallel to BC ?
(ii) In $\triangle ABC$, $AB=AC$ and D, E, F are the mid points of sides BC, CA and AB respectively. How are the straight lines AD and EF related?

(iii) In $\triangle ABC$, D, E, F are the mid-points of sides BC, CA and AB respectively. BE and CF intersect each other at the point O. How are the three points A, O, D related?

(iv) If the centroid, orthocentre, circum-centre and in-centre of a triangle be collinear, what kind of triangle is it?

(v) If the centroid, orthocentre, circum-centre and in-centre of a triangle be the same point, what kind of triangle is it?

(vi) O is the mid-point of the diagonal AC of the parallelogram $ABCD$. What is the measure of $\angle BOD$?

(vii) In $\triangle ABC$, D and E are the mid-points of the sides AB and BC respectively. At which point do the straight lines parallel to BC through the point D and parallel to BA through the point E intersect?

(viii) The length of the diagonal BD of the parallelogram $ABCD$ is 6 cm. P and Q are the centroids of $\triangle ABC$ and $\triangle ADC$ respectively. What is the length of the line segment PQ?

(ix) The median AD of $\triangle ABC$ is produced to the point E, making DE equal to AD. In $\triangle ABC$, the lengths of sides AB, BC and CA are 5 cm, 7 cm and 9 cm respectively. What are the lengths of the two line segments EB and EC?

(x) In $\triangle ABC$, $AB=AC$ and D is the mid-point of BC. The bisectors of $\angle B$ and $\angle C$ intersect each other at the point O. How are the three points A, O, D related?

5. Answer the following questions :

(i) Of all the line segments drawn from any point O, outside the straight line AB to the straight line AB, OP is the least. How are AB and OP related?

(ii) Q is a point on the straight line AB. Two points P, R lie outside the straight line. Both PQ and RQ are perpendiculars to the straight line AB. How are PQ and RQ related?

(iii) $\triangle ABC$ and $\triangle DBC$ are so situated that $AD \parallel BC$. How are the areas of two triangles related?

(iv) $\triangle ABC$ and $\triangle DBC$ are equal in area and the points A and D lie on the same side of BC. How are the straight lines AD and BC related?

(v) Two triangles have the same vertex and their bases lie on the same straight line. How are their heights related?

Miscellaneous Exercise 5

1. Draw perpendicular bisectors of sides AB and BC of $\triangle ABC$. Let their point of intersection be O. Write down the relation between OA, OB and OC.

2. Draw bisectors of $\angle B$ and $\angle C$ of $\triangle ABC$. Let their point of intersection be I. Draw perpendiculars to BC, CA and AB from I. Write down how those perpendicular-lengths are related?

3. Draw a perpendicular to the straight line AB at the point A.

4. Drw a perpendicular bisector of AB. Let the point, where the perpendicular bisector meets AB be C. Draw a semi-circle with C as centre and CA as radius.

5. Draw two line segments a and b, so that $a>b$:

(i) Take a point P and draw a circle with P as centre and radius of length equal to a. Now, draw another concentric circle, whose radius is of length $(a-b)$.

(ii) Take a point Q and draw a circle with Q as centre and radius of length equal to b. Now, draw another concentric circle, whose radius is of length $(a+b)$.

6. Draw a line segment AB of length 9 cm. Bisect the line segment AB. Let O be the mid-point of that line segment. Draw a semi-circle with O as centre and OA as radius. From AB cut off a line segment AC of length 6 cm. Draw a perpendicular to the line segment AB at the point C.

7. Draw a line segment AB, whose length is 9 cm. From AB, cut off a line segment AC of length 6 cm. Bisect AB and AC. Take their respective mid-points as P and Q. Draw a right-angled triangle with AQ as one of the sides and with the hypotenuse whose length is equal to the length of AP.

8. Draw a square $ABCD$ of side 6 cm. Draw an isosceles triangle equal in area to that of the square *(i)* whose base and side BC of the square lie on the same straight line *(ii)* whose base is the diagonal AC of the square.

Miscellaneous Exercise 6

1. Prove that any point on the perpendicular bisector of the line segment AB is equidistant from A and B.

2. The straight lines PQ and RS are two perpendicular bisectors of the line segments AB and BC respectively. If $PQ \parallel RS$, prove that A, B, C are collinear and the length of AC is twice the distance between the straight lines PQ and RS.

3. The straight lines PQ and PR are two perpendicular bisectors of the line segments AB and BC respectively. Prove that $\angle APC = 2\,\angle QPR$.

4. Prove that the angles adjacent to any of the parallel sides of an isosceles trapezium are equal to each other.

5. If two angles adjacent to any of the parallel sides of a trapezium are equal to each other, prove that the trapezium is isosceles.

6. $ABCD$ is an isosceles trapezium and $AB<DC$. DA and CB, when produced, meet at the point P. Prove that $PA=PB$.

7. O is a point interior to $\triangle ABC$, so that $OA=OB=OC$. Using the property that 'the sum of the measures of three angles of a triangle is equal to two right angles', prove that $\angle BOC=2\angle BAC$.

8. In $\triangle ABC$ the median $CO=\frac{1}{2}AB$. Prove that $\angle ACB=1$ right angle.

9. In quadrilateral $ABCD$, $AB=AD$ and $AD \perp DC$ and $AB \perp BC$. Prove that $BC=DC$.

10. A straight line parallel to side BC of $\triangle ABC$ intersects AB and AC at the points P and Q respectively. Prove that $\triangle ABC$ and $\triangle APQ$ are equiangular.

(You know that two triangles are equiangular if three angles of one triangle are equal to those of the other.)

11. A straight line parallel to side QR of $\triangle PQR$ intersects PQ and PR at the points X and Y respectively. Join Q, Y and R, X. QY and RX intersect each other at the point O. Prove that *(i)* $\triangle PYQ = \triangle PXR$ *(ii)* $\triangle QOX = \triangle ROY$.

12. In $\triangle ABC$, $\angle A$ is a right angle. AD is perpendicular to the side BC. Write down the two angles complement to $\angle DAC$. How are these two angles related? Write down the two angles complement to $\angle ACD$; state the relation between them. Hence prove that $\triangle ABD$ and $\triangle CAD$ are equiangular.

13. Any two of three straight lines are parallel to each other. If the intercepts of each of any transversal are equal, prove that the three straight lines are parallel.

14. In parallelogram $ABCD$, P and Q are the mid-points of sides AB and AD respectively. R and S are centroids of $\triangle ADC$ and $\triangle ABC$ respectively. What kind of quadrilateral $PQRS$ is? Give reasons in support of your answer.

15. D is any point on side AC of $\triangle ABC$. E, F, G, H are the mid-points of AD, DC, BC and AB respectively. Prove that EG and FH bisect each other.

16. AB and CD are two parallel line segments. E is the mid-point of CD. AC and BE, when produced, intersect each other at the point F and AE and BD, when produced, intersect each other at the point G. Prove that $FG \parallel AB$.

17. D is any point on side BC of the triangle ABC. The straight lines drawn through the point D parallel to CA and BA intersect BA and CA at the points E and F respectively. Prove that $\triangle FBD = \triangle EDC$.

18. $\triangle ABC$ is an isosceles triangle, right-angled at the point B. The bisector of $\angle BAC$ intersects BC at the point D. DE is drawn perpendicular to AC from the point D. Prove that $\triangle DEC$ is also a right-angled isosceles triangle.

19. $ABCD$ is a square. Points E, F, G, H are so taken on AB, BC, CD, DA respectivelyy that $AE = BF = CG = DH$. Prove that $EFGH$ is a square.

20. Prove that the end points of a line segment are equidistant from any point on the perpendicular bisector of the line segment.

21. Prove that any point on the bisector of the angle included by two intersecting straight lines is equidistant from the two straight lines.

22. Prove that the perpendicular bisectors of four-sides of an isosceles trapezium are concurrent.

23. Two triangles have their two sides equal and also have the angles opposite to one pair of corresponding sides equal. Prove that the angles opposite to other pair of corresponding sides are either equal or supplementary. Further,prove that they must be equal if they are either acute or obtuse. How are the two triangles related in that case ?

CHAPTER-4

CIRCLE

4.1. You are already familiar with what a circle is. You also have an idea of a circle. Let us now define it.

Definition : A plane figure formed by all the points equidistant from a fixed point is a circle. Fig. 89 is the figure of a circle. You have also learnt, in class VI, the meaning of some terms associated with circle. let us take up them once again.

Centre : The fixed point from which the points on the circle are equidistant is called the *centre* of the circle. In Fig. 89, O is the centre of the circle.

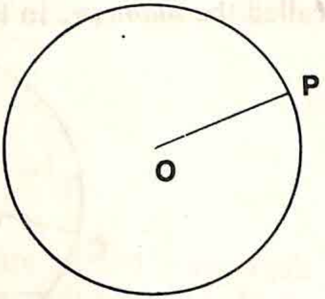


Fig. 89

[*Remark :* You have come to know of the interior of region and the boundary line of any plane figure in Class IX. The centre of the circle is in the interior of the circle. The curve bounding the circle is the boundary of the circular figure.]

Circumference : The length of the boundary of the circle is called the *circumference* of the circle.

Radius. The line segment drawn from the centre to any point on the circle is called the *radius* of the circle. In Fig. 89, P is any point on the circle. OP is a radius of circle.

We can now say, from the definition of the circle, that all radii of a circle are equal in length.

[*Remark* : The point whose distance from the centre is less than the radius is in the interior of the circle, the point whose distance is equal to the radius lies on the boundary and the point whose distance is greater than the radius does not lie in the interior or the boundary of the circle].

Chord : The line segment joining any two points on a circle is called the *chord* of the circle. In Fig. 90, *AB* is a chord of the circle.

Diameter : The chord passing through the centre is called the *diameter*. In Fig. 90, *CD* is a diameter.

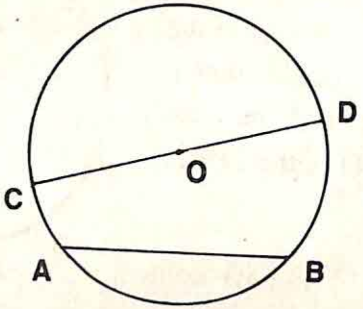


Fig 90

Observe that the length of a diameter is twice the length of a radius.

Arc of a circle : The part of the curve that forms the circle is called the arc of a circle. In Fig. 91, *PQ* is an arc of the circle.

[*Remark* : If the arc is smaller than half the boundary, it is called minor arc and if greater than half the boundary, it is the major arc.]

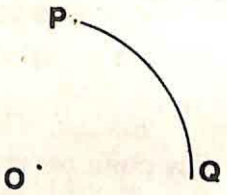


Fig. 91

Segment of a Circle : The closed figure formed by a chord of a circle and an arc is called the *segment of a circle*.

In Fig. 92, *ABC* is a segment of a circle. It is formed by the chord *AB* and the arc of the circle *ACB*.

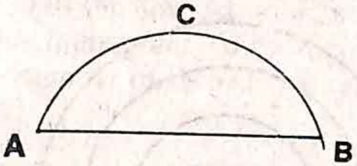


Fig. 92

Semi-Circle : The closed figure formed by a diameter and half the boundary is called the *semi-circle.* In Fig. 93, *ACB* is a semi-circle. It is formed by the diameter *AB* and the arc *ACB.*

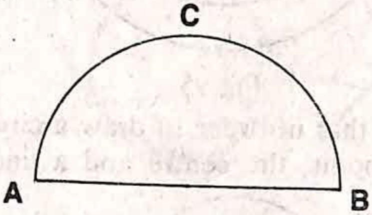


Fig. 93

Sector of Circle : The close figure formed by two radii and an arc of a circle is called the *sector of a circle.* In Fig. 94, *AOB* is a sector of a circle.

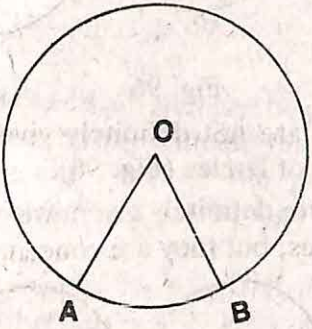


Fig. 94

Concentric circle : Circles drawn with the same centre (and different radii) are called *Concentric circles.* In

Fig. 95, all the circles are drawn with the point O as centre. These circles are concentric circles.

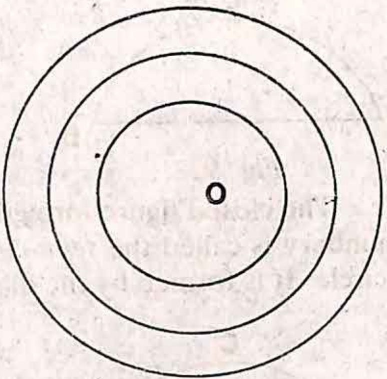


Fig.95

It is clear that in order to draw a circle, we need to have a fixed point, the centre and a line segment, the radius. If these are not definitely given we can have an infinite number of circles (Fig. 96).

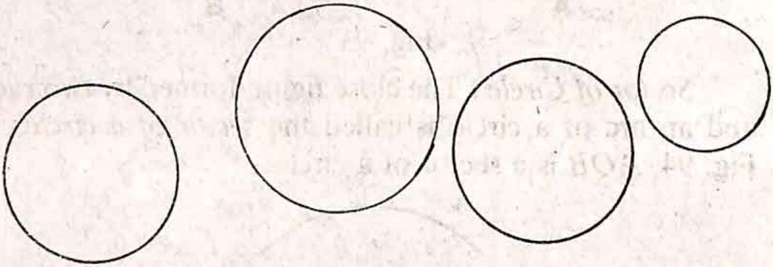


Fig. 96

With a centre definitely given, we can draw an infinite number of circles, but they are concentric circle (Fig. 95).

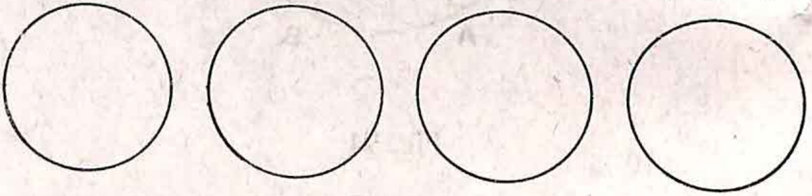


Fig. 97

With the radius being definitely given. we can draw an infinite number of circles. The lengths of the radii of those circles are equal (Fig. 97)

These circles are called congruent circles.

Now let us see how many circles can be drawn passing through a definite point. In this case also we observe that an infinite number of such circles can be drawn (Fig. 98).

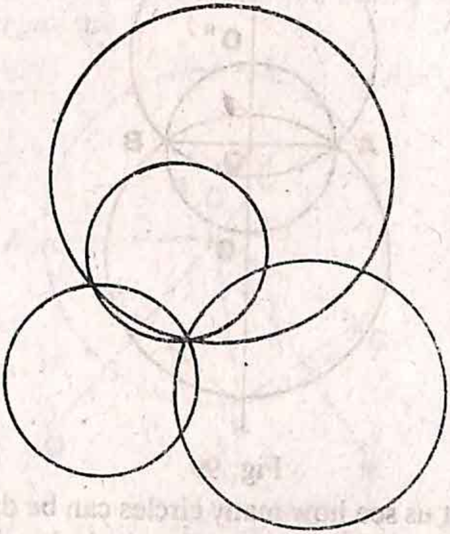


Fig. 98

In fact, all the circles that can be drawn with any point on the plane as centre (excluding that definite point) and taking the distance between the point taken and the definite point as radius, pass through that definite point.

How many circle do we get if the circles are to be drawn passing through two definite points?

Let *A* and *B* be two definite points. Then the centres of the circles are so situated that each of them is equidistant from *A* and *B*. We have already seen that any point on the perpendicular bisector of the line segment *AB* is equidistant from *A* and B. Therefore, any circle drawn with any point on the perpendicular bisector of *AB* as centre and the line segment joining that point and either the point A or the point B as radius passes through A and B.

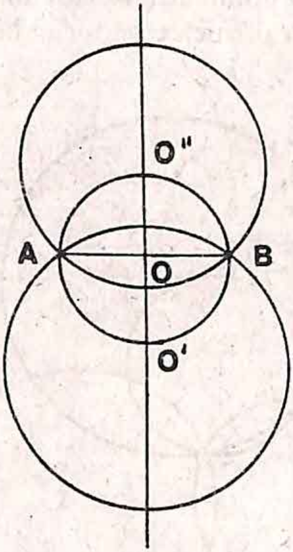


Fig. 99

Now let us see how many circles can be drawn passing through three definite points, particularly when they are non-collinear.

4·2. We will find that one and only circle can be drawn through three non-collinear points. This property will be established in the next theorem.

Theorem 34

One and only one circle can be drawn through three non-collinear points.

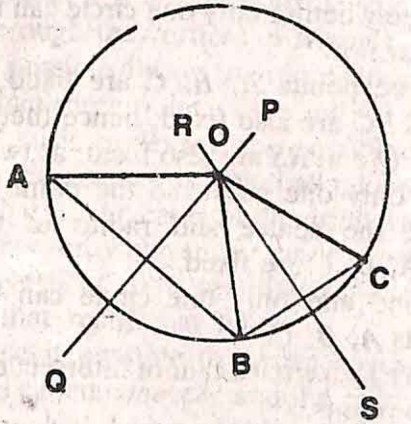


Fig. 100

Let *A*, *B*, *C* be three non-collinear points.

To prove that one and only one circle can be drawn through those three points.

Construction : *A*, *B* and *B*, *C* are joined. Perpendicular bisectors of *AB* and *BC* are drawn. They are *PQ* and *RS* respectively. *PQ* and *RS* will intersect each other, as *A*, *B*, *C* are non-collinear. Let them intersect each other at the point *O*, *A*, *B* and *C* are joined with the point *O*.

Proof : ∵ the point *O* lies on the perpendicular bisector of *PQ*,

∴ the point *O* is equidistant from *A* and *B*;
that is, $OA=OB$.

and DC parallel to each other and the oblique sides equal; that is, $DA=CB$.

To Prove that A, B, C, D are concyclic.

Construction : The perpendicular bisector PQ of side DC is drawn. That perpendicular bisector will also be perpendicular bisector of side AB. Let PQ intersect DC and AB at the points R and S respectively. Let us draw perpendicular bisectors LM of AD. LM intersects PQ at the point O and AD at the point N.

The points A, B, C, D are joined with the point O.

Proof : $\because$ O lies on the perpendicular bisector of DC,

$\therefore OC=OD$

Similarly, $OD=OA$

and $OA=OB$

$\therefore OC=OD=OA=OB$

$\therefore$ the circle drawn with the point O as centre and OC as radius passes through the points D, A, B.

That is A, B, C, D are concyclic.

[*Remark* : We shall use this property at a later stage to prove a theorem.]

4.4. Theorems relating Centre and Chord :

Theorem 35

A straight line, drawn from the centre of a circle to bisect a chord which is not a diameter, is at right angles to the chord.

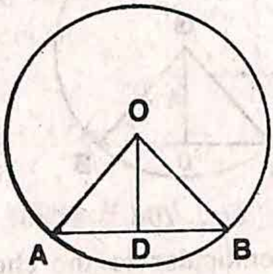


Fig. 103

Let the straight line OD bisect the chord AB of the circle with centre O at the point D.

To prove that OD is perpendicular to AB.

Construction : O, A and O, B are joined.

Proof : In $\triangle OAD$ and $\triangle OBD$,
$OA=OB$ (radii of the same circle),
OD is common and
$AD=DB$ (by hypothesis)

$\therefore \triangle AOD \cong \triangle OBD$

$\therefore \angle ODA = \angle ODB$

$\therefore$ DO stading on AB makes equal angles.

$\therefore$ OD is perpendicular to AB.

[*Remark* : $\triangle OAB$ is an isosceles triangle.

$\therefore \angle OAB = \angle OBA$. Now the theorem could have been proved by SAS congruence property. Try for yourself.]

Theorem 36

The perpendicular drawn from the centre of a circle to a chord which is not a diameter bisects the chord.

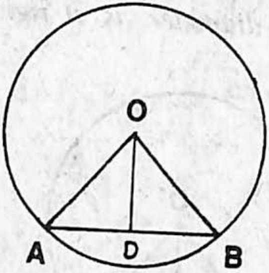


Fig. 104

Let OD be perpendicular to the chord AB of the circle with centre O.

To prove that OD bisects AB; that is, $AD=DB$.

Construction : O, A and O, B are joined.

Proof : $\because OD \perp AB$,

$\triangle ODA$ and $\triangle ODB$ are right-angled triangles.

$\because$ in right-angled triangle $\triangle ODA$ and $\triangle ODB$, hypotenuse OA=hypotenuse OB (radii of the same circle)

OD is common

$\therefore \triangle ODA \cong \triangle ODE$,

$AD=BD$ (corresponding sides).

[*Remark* : In this case also, as $\triangle OAB$ is an isosceles triangle, $\angle OAB = \angle OBA$. Now, using the AAS congruence property this theorem can be proved, Try for this.]

Corollary 1 : The perpendicular bisector of any chord of a circle passes through the centre of the circle.

[*Remark* : If two chords of a circle or three points on the circle are given we can find the centre and radius of the circle. This property will be used at the time of construction of circum-circle of a triangle.]

Corollary 2 : a straight line cannot intersect a circle at more than two points.

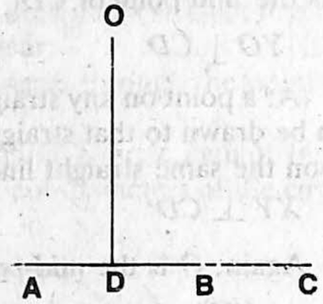


Fig. 105

If possible, let the straight line intersect the circle with centre O at the points A, B and C. Perpendicular is drawn to the straight line from the point O.

$\therefore$ $AD=DB$ and $AD=DC$,

that is $DB=DC$,

which is impossible, unless B and C are not coincident points.

Some Applications :

1. Two circles with centres X and Y intersect each other at the points C and D. Prove that XY is the perpendicular bisector of the common chord CD.

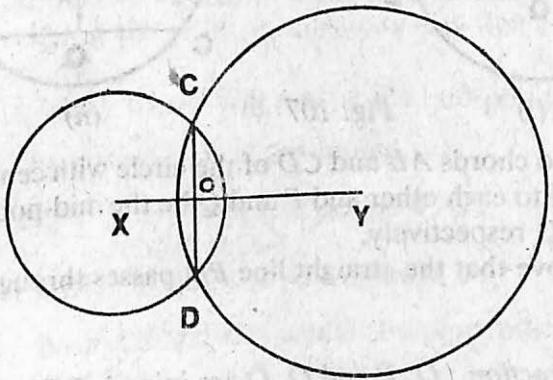


Fig. 106

Construction : Perpendicular XO is drawn to CD from the point X. O, Y are joined.

Proof : CD is the chord of the circle with centre X and $XO \perp CD$.

$\therefore$ O is the mid-point of CD.

Again, CD is the chord of the circle with centre Y and O is the mid-point of CD,

$\therefore\ YO \perp CD$.

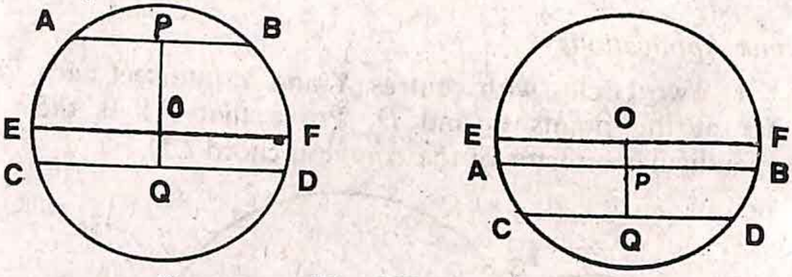
At a point on any straight line only one perpendicular can be drawn to that straight line; therefore, XO and OY lie on the same straight line, so that

$XY \perp CD$

Again, O is the mid-point of CD,

$\because$ XY is the perpendicular bisector of CD.

2. Two chords of a circle are parallel to each other. Prove that the straight line joining their mid-points passes through the centre of the circle.



(i) Fig. 107 (ii)

Let two chords AB and CD of the circle with centre O be parallel to each other and P and Q be the mid-points of AB and CD respectively.

To prove that the straight line PQ passes through the point O.

Construction : O, P and O, Q are joined. EF is drawn parallel to AB and CD through the point O.

Proof : $\because$ P is the mid-point of AB,

$\therefore$ $OP \perp AB$.

$\because$ $AB \parallel EF$,

$\therefore$ $OP \perp EF$.

Similarly, $OQ \perp EF$.

$\therefore$ OP and OQ are both perpendiculars to EF. But since only one perpendicular can be drawn to a straight line at a point on it, P, O, Q are collinear.

$\therefore$ The straight line PQ passes through the point O

3. Two intersecting chords of a circle cannot bisect each other, if both of them are not diameters of the circle. Prove this.

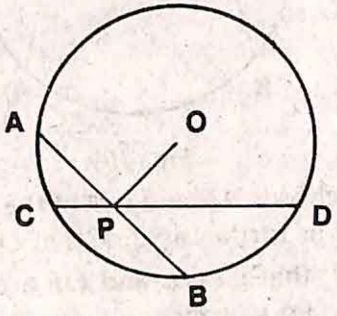


Fig. 108

Let the two chords of a circle with centre O intersect each other at the point P, in such a way that P is the mid-point of AB.

To prove that P will not be the mid-point of CD.

Construction : O, P are joined.

Proof : $\because$ P is the mid-point of AB,
$\therefore$ $OP \perp AB$

$\therefore$ both AB and CD pass through P,

$\because$ both AB and CD cannot be perpendicular to OP at the point P.

$\because$ AB is perpendicular to OP,
$\therefore$ CD is not perpendicular to OP.

Now, since the line segment joining the mid-point of a chord and the centre of the circle is perpendicular to the chord, P is not the mid-point of CD.

[If both of them are diameters, then by definition, they will surely bisect each other.]

4. Prove that two equal chords of a circle are equidistant from the centre.

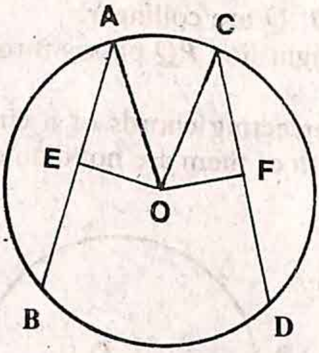


Fig. 109

Let two chords AB and CD of the circle with centre O be equal. Let us further assume that $OE \perp AB$ and $OF \perp CD$: that is, OE and OF are respectively the distances of AB and CD from the centre O.

To prove that $OE = OF$.

Construction : O, A and O, C are joined.

Proof : $\because OE \perp AB$ and $OF \perp$ CD,

$\therefore \triangle OEA$ and $\triangle OFC$ are both right-angled triangles.

For the same reason E and F are the mid-points of AB and CD respectively; that is,

$AE = \frac{1}{2} AB$ and $CF = \frac{1}{2} CD$.

$\because AB = CD$,

$\therefore AE = CF$,

Now, in right-angled triangles $\triangle OEA$ and $\triangle OFC$, hypotenuse OA = hypotenuse OC (radii of same circle) and $AE = CF$ (proved)

$\therefore \triangle OEA \cong \triangle OFC$,

$\therefore OE = OF$ (corresponding sides).

Exercise 16

1. The length of the radius of a circle with centre O is 5 cm and the length of the chord AB is 8 cm. What is the distance of the chord AB from the point O ? Give reasons.

2. The length of the diameter of a circle with centre O is 26 cm. The distance of the chord PQ from the point O is 5 cm. What is the length of the chord PQ ? Give reasons.

3. The length of the chord PQ of a circle with centre O is 4 cm and the distance of PQ from the point O is 2·1 cm. What is the length of the diameter of the circle? Give reasons.

4. The lengths of two parallel chords of a circle of radius 10 cm are 16 cm and 12 cm. What is the distance between the two chords, if they lie (i) on the same side (ii) on the opposite side of the centre? Give reasons.

5. The length of a chord of a circle is 48 cm and the distance of that chord from the centre is 7 cm. What is length of that chord whose distance from the centre of that circle is 20 cm? Give reasons.

6. In Fig. 110 the point O is the centre of the circle. $OP \perp AB$. $AB=6$ cm, $PC=2$ cm; what is the length of the radius of the circle? Give reasons.

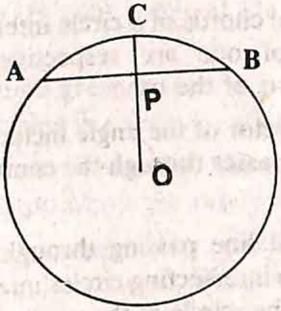


Fig. 110

7. A straight line intersects one of the two concentric circles at the points A and B and the other at the points C and D. Prove that $AC=DB$.

8. Two circles with centres P and Q intersect each other at the points A and B. Prove that PQ bisects $\angle APB$ and $\angle AQB$.

9. H, K are centres of two equal circles and they intersect each other at the points R and S. A straight line is drawn parallel to HK through the point R, intersects the two circles at the points C and D. Prove that $CD=2\ HK$.

10. Two circles with centres X and Y intersect each other at the points A, B. The point A is joined with S, the mid-point of XY, The perpendicular to SA through the point A intersects the circles at the points P and Q. Prove that $PA=AQ$.

11. Prove that two chords of a circle equidistant from the centre are equal.

12. Two intersecting chords of a circle make equal angles with the diameter passing through the point of intersection of the chords. Prove that two chords are equal.

13. PQ, QR, RS, ST are chords of a circle. If $PQ=QR=RS=ST$, then prove that $PR=QS=RT$.

14. Two chords AB and AC of a circle are equal. Prove that the bisector of $\angle BAC$ will pass through tthe centre.

15. PQ is a fixed chord of a circle and AB is any diameter. Prove that the sum (or difference) of the perpendicular-lengths of PQ from the points A and B will remain constant for any position of AB.

16. Two equal chords of a circle intersect each other. Prove that two parts of one are respectively equal to the two corresponding parts of the other.

17. If the bisector of the angle included by two intersecting chords of a circle passes through the centre, then prove that two chords are equal.

18. A straight line passing through one of the points of intersection of two intersecting circles intersects one circle at the point P and the other circle at the point Q. If the straight line is parallel to the line segment joining the two centres, prove that PQ will be greatest.

4.5. Theorems related to Circle and Angle

Before we proceed with the theorems, we need to have some definitions.

Definitions : 1. *Angle at the centre :* The angle formed by joining the centre with two points on a circle is called *angle at the centre.* The angle subtended by the arc is called the angle at the centre standing on that arc.

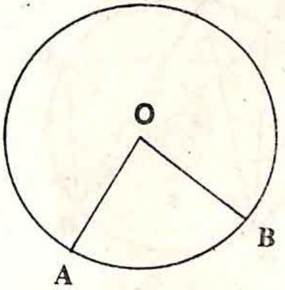


Fig. 111

In fig, 111, $\angle AOB$ or the reflex $\angle AOB$ are angle at the centre.

$\angle AOB$ is the angle at the centre standing on the minor arc AB and reflex $\angle AOB$ is the angle at the centre standing on the major arc AB.

[*Remarks :* An angle greater than two right angles but less than four right angles is called a *reflex angle.*]

2. *Angle at the circumference.* If the end points of an arc are joined with any point on the other arc, the angle so formed is called *angle at the circumference.* The angle subtended by the arc is called the angle at the circumference standing on that arc.

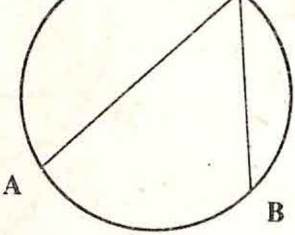


Fig. 112

In fig. 112, $\angle ACB$ is the angle at the circumference and that angle is standing on the minor arc AB.

[*Remarks* : (1) Although only one angle at the centre stands on an arc, the number of angles at the circumference standing on the same may be many.

For example, in Fig. 113, ∠ *AOB* is the only angle at the centre standing on the arc *APB;* but ∠ *ACB*, ∠ *ADB*, ∠*AEB* etc, are the angles at the circumference standing on the same arc.

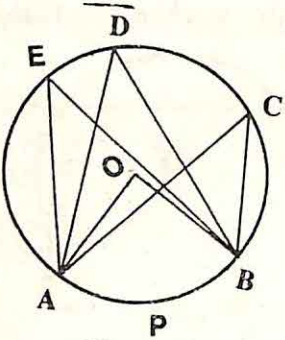


Fig. 113

(2) Since here minor arc or major arc is not mentioned, therefore the arc is mentioned taking a point on the arc under consideration.]

3. *Angle in the segment of a circle :* The angle subtended by the chord of the segment of a circle at any point on the arc of that segment of the circle is called the *angle in the segment of a circle.*

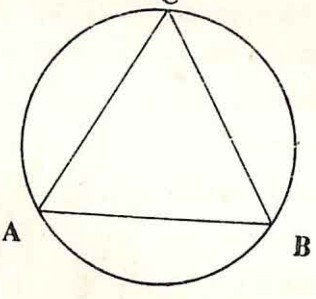


Fig. 114

In fig. 114, ∠ *ACB* is the angle in the segment *ACB* of the circle.

[*Remarks :* (1) The segment of the circle formed by the chord AB and the arc ACB is called the segment ACB of the circle.

(2) Observe, the angle in the segment ACB of the circle may also be called the angle at the circumference standing on the minor arc AB.]

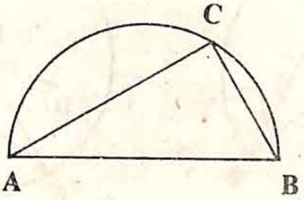


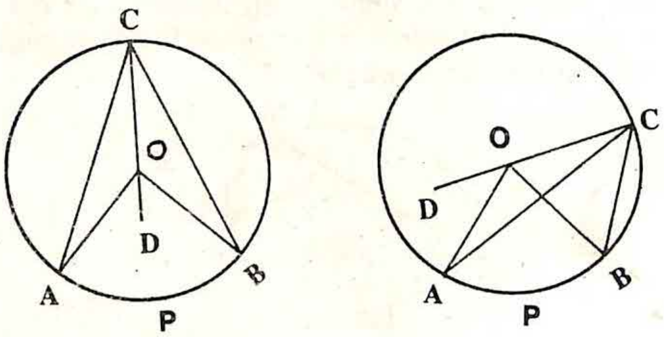
Fig. 115

4. *Angle in a semi-circle :* The angle subtended by the diameter of a semi-circle at any point on the semi-boundary of the circle is called the *angle in a semi-circle.*

In fig. 115, $\angle ACB$ is an angle in the semi-circle.

Theorem 37

The angle at the centre of a circle is double the angle at the circumference standing on the same arc.



(i) *Fig. 116* (ii)

Let $\angle AOB$ is the angle at the centre standing on the arc APB of the circle with centre O and $\angle ACB$ is the angle at the circumference standing on the same arc.

To prove that $\angle AOB = 2 \angle ACB$.

Construction : C, O are joined and CO is produced to any point D.

Proof : In $\triangle AOC$,

$OA = OC$ (radii of the same circle), $\therefore \angle OCA = \angle OAC$.
Again, since side CO of $\triangle AOC$ is produced to the point D

$$\therefore \text{ Exterior } \angle AOD = \angle OAC + \angle OCA$$
$$= 2 \angle OCA.$$

Similarly, from $\triangle BOC$, we get
$\angle BOD = 2 \angle OCB$

$\therefore$ From fig. No, *(i)* we get

$$\angle AOB = \angle AOD + \angle BOD$$
$$= 2 (\angle OCA + \angle OCB)$$
$$= 2 \angle ACB$$

and from Fig. No, *(ii)* we get
$\angle AOB = \angle BOD - \angle AOD$

$= 2\,(\angle OCB - \angle OCA)$

$= 2\,\angle ACB$

Some applications :

1. Fig. 117, O is the centre of the circle, $\angle OAB = 30°$, $\angle ABC = 120°$, $\angle BCO = y°$ and $\angle COA = x°$. Find the values of x and y; Give reasons.

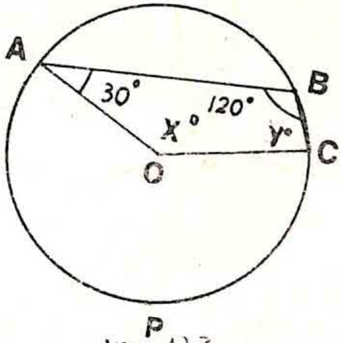


Fig: 117

Answer : $x = 120$, $y = 90$

Proof : ∵ Reflex $\angle AOC$, the angle at the centre and $\angle ABC$, the angle at the circumference stand on the same arc APC,

$$\therefore \text{Reflex } \angle AOC = 2\,\angle ABC = 2 \times 120° = 240°$$

$$\therefore \angle COA = 360° - 240° = 120°,$$

$$\therefore x = 120.$$

Again, $\because \angle OAB + \angle ABC + \angle BCO + \angle COA = 360°$

$$\therefore 30° + 120° + y° + 120° = 360°$$

or, $y° + 270° = 360°$

or, $y° = 360° - 270° = 90°$.

$$\therefore y = 90.$$

2. Two equal circles intersect each other at the points A and B. A straight line through the point A intersects one of the circles at the point P and the other circle at the point Q. Prove that $BP = BQ$.

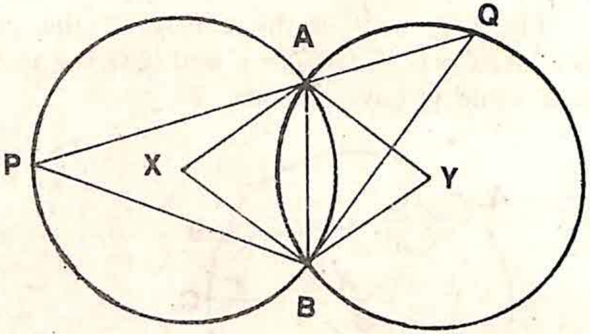


Fig. 118

Construction: Let X, Y be respectively the centres of the first and the second circle. A, B; A, X; B, X; A, Y and B, Y are joined.

Proof: In $\triangle AXB$, $\triangle AYB$,

$AX = AY$ (radii of the same circle),

$BX = BY$ (radii of the same circle),

AB is common.

$\therefore \triangle AXB \cong \triangle AYB$

$\therefore \angle AXB = \angle AYB$ (corresponding angles)

But, $\because \angle APB$ is the angle at the circumference and $\angle AXB$ is the angle at the centre standing on the same arc,

$\therefore \angle APB = \frac{1}{2} \angle AXB$

similarly, $\angle AQB = \frac{1}{2} \angle AYB$

$\therefore \angle APB = \angle AQB$ ($\because \angle AXB = \angle AYB$)

that is, $\angle QPB = \angle PQB$

In $\triangle PQB$, $\angle QPB = \angle PQB$

$\therefore BP = BQ$.

3. ABC is an isosceles triangle, with sides $AB = AC$. $\triangle DBC$ lies on that side of BC in which $\triangle ABC$ lies and

$\angle BAC = 2\angle BDC$. Prove that the point D lies on the circle drawn with A as centre and AB as radius.

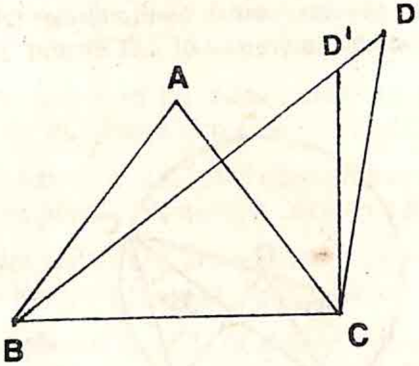


Fig. 119

Let us assume that D does not lie on that circle Then let us suppose that the circle intersects BD at the point D'.

$\because$ $\angle BAC$ and $\angle BD'C$ are respectively angle at the centre and angle at the circumference standing on the same arc,

$\therefore \angle BAC = 2\angle BD'C$

But $\angle BAC = 2\angle BDC$ (by hypothesis)

$\therefore \angle BD'C = \angle BDC$

But this is impossible if the points D and D' do not coincide, since in $\triangle DD'C$,

The exterior $\angle BD'C >$ the interior opposite $\angle BDC$.
$\therefore$ *the point* D lies on the circle.

Exercise 17

1. In fig. 120, O is the centre of the circle; $\angle AOD = 40°$, $\angle ACB = 35°$. Find the measures of $\angle BCO$ and $\angle BOD$. Give reasons.

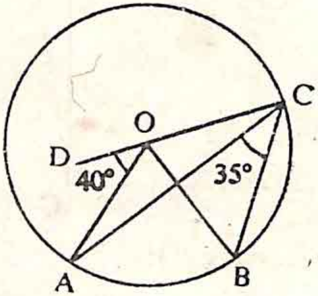


Fig. 120

2. In Fig. 121, O is the centre of the circle. If $\angle APB = 80°$, find the sum of the measures of $\angle AOB$ and $\angle COD$. Give reasons.

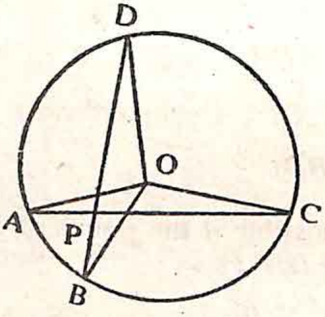


Fig. 121

3. Two circles with centres C and D intersect each other at the points A and B. A straight line through the point A intersects the circle with centre C at the point P and the circle with centre D at the point Q. Prove that

(i) $\angle PBQ = \angle CAD$, (ii) $\angle BPC = \angle BQD$.

4. OA is a radius of a circle with centre O; AQ is a chord and C is a point on the circle. A circle passing through the points O, A, C intersects AQ at the point P. Prove that $CP = PQ$.

5. Two equal circles pass through the centres of each other. The two circles intersect each other at the points A and B. A straight line through the point A intersects the two circles at the points C and D. Prove that $\triangle BCD$ is equilateral.

6. S is the centre of the circum-circle of $\triangle ABC$. AD is perpendicular to BC. Prove that $\angle BAD = \angle SAC$.

7. Two chords AB and CD of a circle with centre O intersect each other at the point P. Prove that $\angle AOD + \angle BOC = 2\angle BPC$.

If $\angle AOD$ and $\angle BOC$ are supplementary to each other, prove that the two chords are at right angles.

8. Two chords AB and CD of a circle with centre O, when produced, intersect each other at the point P. Prove that

$$\angle AOC - \angle BOD = 2\angle BPC.$$

9. PR is a diameter of a circle and PQ is a chord. Prove that the radius OR, parallel to PQ, bisects the arc QR.

Theorem 38

Angles in the same segment of a circle are equal.

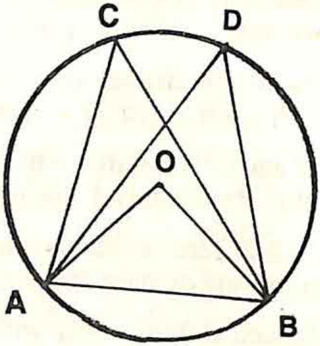


Fig. 122

Let $\angle ACB$ and $\angle ADB$ be any two angles in the segment $ACDB$ of the circle.

To prove that all angles in the segment $ACDB$ of the circle are equal.

Since $\angle ACB$ and $\angle ADB$ are any two angles on that segment of the circle, the theorem will be proved if it can be shown that $\angle ACB$ and $\angle ADB$ are equal to each other.

Construction: Let O be the centre of the circle. O, A and O, B are joined.

Proof: $\angle AOB$ is the angle at the centre and, $\angle ACB$ and $\angle ADB$ are angles at the circumference standing on the same arc APB,

$\therefore\ \angle AOB = 2\angle ACB$

and $\angle AOB = 2\,\angle ADB$

$\therefore\ 2\angle ACB = 2\angle ADB$

$\therefore\ \angle ACB = \angle ADB$

[*Remark:* We could have stated the general enunciation of theorem as below:

All angles at the circumference of a circle standing on the same arc are equal:]

2. Since angle in a segment is involved the, AB must be drawn in the figure.

3. If the general enunciation is given according to (1), AB is not to be drawn.]

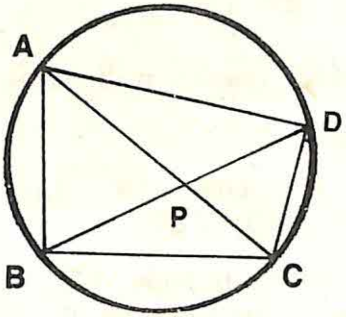


Fig. 123

Some Applications:

1. In Fig. 123, $\angle BDC = 50°$, $\angle APB = 65°$, $\angle CBD = 28°$; find the measures of $\angle ADB$, $\angle ABD$, $\angle BAC$, $\angle ACB$, $\angle CAD$ and $\angle ACD$.

$\angle BAC = \angle BDC$ (angles in the same segment of the circle)

$= 50°$

$\angle CAD = \angle CBD$ (angles in the same segment of the circle)

$= 28°$

$\angle APB$ is the exterior angle of $\triangle BPC$ and $\angle PBC$ and $\angle PCB$ are interior opposite angles.

$$\therefore \angle APB = \angle PBC + \angle PCB$$
$$= \angle CBD + \angle ACB.$$
$$\therefore 65° = 28° + \angle ACB$$
$$\therefore \angle ACB = 65° - 28° = 37°$$

$\angle ADB = \angle ACB$ (angles in the same segment of the circle)

$= 37°$

Again, in $\triangle APB$,

$$\begin{aligned} \angle BAP + \angle ABP + \angle APB &= 180°; \\ \text{that is, } \angle BAC + \angle ABC + \angle APB &= 180°, \\ \text{or, } 50° + \angle ABD + 65° &= 180°, \\ \text{or, } \angle ABD + 115° &= 180°, \\ \text{or, } \angle ABD = 180° - 115° &= 65°. \end{aligned}$$

$\angle ACD = \angle ABD$ (angles in the same segment of the circle)

$= 65°$

$\therefore$ $\angle ADB = 37°$, $\angle ABD = 65°$, $\angle BAC = 50°$, $\angle ACB = 37°$, $\angle CAD = 28°$, $\angle ACD = 65°$.

2. The equilateral triangle ABC is inscribed in a circle. P is a point on the minor arc BC. Prove that $PA = PB + PC$.

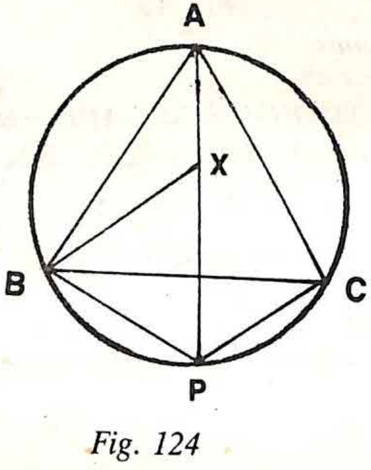


Fig. 124

Construction: PX is cut off from PA making equal to PB. B, X are joined.

Proof: in $\triangle PBX$,

$\because PB = PX$, $\therefore \angle PXB = \angle PBX$

Again, $\angle BPX$

$= \angle ACB$ (angles in the same segment of the circle)

$= 60°$ ($\because$ $\triangle ABC$ is equilateral;)

$\therefore \angle PXB + \angle PBX = 120°$

$\therefore \angle PXB = \angle PBX = 60°$

$\therefore$ $\triangle PBX$ is equilateral.

$\therefore \angle PBX = \angle CBA$ (each of them is 60°)

$$\begin{aligned}\angle PBC &= \angle PBX - \angle CBX\\ &= \angle CBA - \angle CBX\\ &= \angle XBA\end{aligned}$$

Now, in $\triangle AXB$ and $\triangle CPB$,

$AB = BC$ ($\because$ $\triangle ABC$ is equilateral)

$BX = BP$ ($\because$ $\triangle PBX$ is equilateral)

included $\angle XBA$ = included $\angle PBC$ (Proved),

$\therefore \triangle AXB \cong \triangle CPB$

$\therefore AX = PC$ (corresponding sides).

$$\begin{aligned}\therefore \mathrm{PA} &= \mathrm{PX} + \mathrm{XA}\\ &= \mathrm{PB} + \mathrm{PC}\end{aligned}$$

[*Remark:* Since $PA = PB + PC$ is to be proved, PX has been cut off making equal to PB, so that the property will be established if it can be proved that $XA = PC$.]

Exercise 18

1. In Fig. 125, $\triangle DBA = 40°$; $\angle BAC = 60°$ and $\angle CAD = 20°$; find the measures of $\angle DCA$ and $\angle BCA$. Observe also what the sum of the measures of $\angle BAD$ and $\angle DCB$ will be.

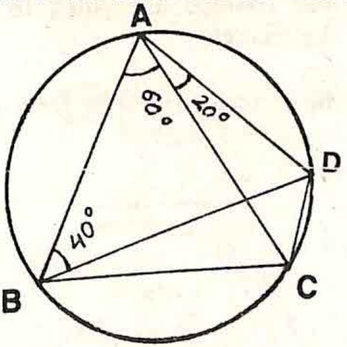


Fig. 125

2. O is the orthocentre of the triangle ABC and perpendicular AD drawn to BC, when produced, intersects the circumcircle of $\triangle ABC$ at the point G. Prove that $OD = DG$.

3. I is the centre of the in-circle of $\triangle ABC$. AI produced intersects the circum-circle of that triangle at the point P. Prove that $PB = PC = PI$.

4. $ABCD$ is a cyclic quadrilateral, so that AC bisects $\angle BAD$. AD is produced to the point E, so that $DE = AB$. Prove that $CE = CA$.

5. Two circles intersect each other at the points P and Q respectively. Through the point P two straight lines are drawn. They intersect one of the circles at the points A and B, and the other circle at the points C and D respectively. Prove that $\angle AQC = \angle BQD$.

6. One of the two circles passes through the point O, the centre of the other. The two circles intersect each other at the points A and B. A straight line through the point A intersects the circle passing through the point O at the point P and the circle with centre O at the point R. Join P, B and R, B and prove that $PR = PB$.

7. Two chords AB and CD of a circle are perpendicular to each other. If the perpendicular drawn to AD from the point O, the point of intersection of those two chords is produced to meet BC at the point E, prove that E is the mid-point of BC.

8. One angle of an isosceles triangle is 120°; prove that the equal sides of that triangle are equal to the radius of the circum-circle of the triangle.

9. In the cyclic quadrilateral $ABCD$, $AB = DC$. Prove that $AC = BD$.

Theorem 39

Angle in the semi-circle is a right angle.

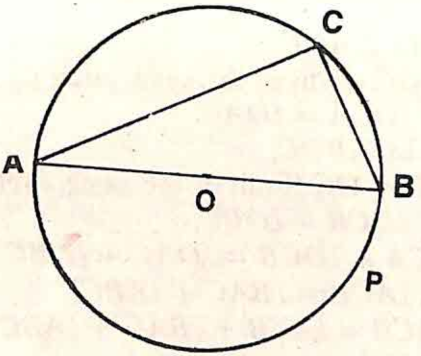


Fig. 126

Let, in the circle with centre O, $\angle ACB$ is an angle in the semi-circle.

To prove that $\angle ACB = 1$ right angle.

Proof: $\angle AOB$ is the angle at the centre and $\angle ACB$ is the angle at the circumference standing on the same arc APB,

$\therefore \angle AOB = 2\angle ACB$.

But AOB is a line segment,

$\therefore \angle AOB = 2$ right angles

that is, $2\angle ACB = 2$ right angles,

$\therefore \angle ACB = 1$ right angle.

Alternative Proof:

Let B be a semi-circle, where O is the centre of the complete circle and $\angle ACB$ is an angle in the semi-circle.

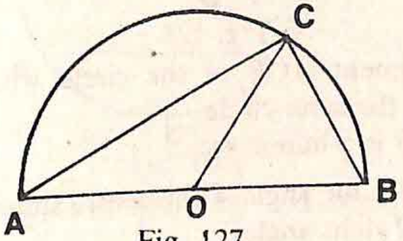


Fig. 127

To prove that
$\angle ACB = 1$ right angle.

Construction: C, O are joined.

Proof: In $\triangle AOC$,
$OA = OC$ (radii of the same circle)
$\therefore \angle OCA = \angle OAC.$
Again, in $\triangle BOC$,
$OB = OC$ (radii of the same circle)
$\therefore \angle OCB = \angle OBC,$
$\therefore \angle OCA + \angle OCB = \angle OAC + \angle OBC$
that is, $\angle ACB = \angle BAC + \angle ABC,$
$\therefore 2\angle ACB = \angle ACB + \angle BAC + \angle ABC$
$= 180°$
$\therefore \angle ACB = 90° = 1$ right angle.

[*Remark:* Since in the alternative proof, the other arc of the circle has not been used, therefore, only the semi-circle has been drawn.]

Corollary 1: The angle in the segment of a circle which is greater than a semi-circle is an acute angle.

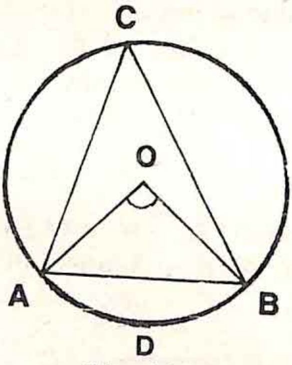


Fig. 128

The segment ACB of the circle with centre O is greater than the semi-circle.

$\therefore$ ADB is a minor arc.

$\therefore \angle AOB$, the angle at the centre standing on that arc is less than 2 right angles.

∵ $\angle ACB$ is an angle at the circumference also standing on the same arc,

∴ $\angle ACB$ is less than 1 right angle, that is, $\angle ACB$ is an acute angle.

Corollary 2: The angle in the segment of a circle which is less than a semi-circle is an obtuse angle.

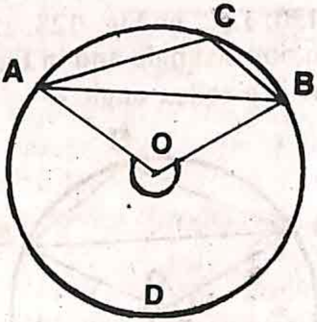


Fig. 129

The segment ACB is less than the semi-circle.

∴ ADB is a major arc.

∴ The reflex $\angle AOB$, the angle at the centre standing on that arc is greater than 2 right angles.

∵ $\angle ACB$ is an angle at the circumference also standing on that same arc,

∴ $\angle ACB$ is greater than 1 right angle.

∴ $\angle ACB$ is an obtuse angle.

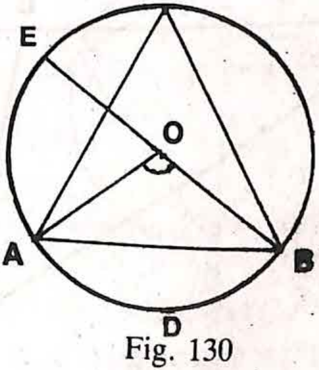


Fig. 130

[*Remark:* Fig. 130 and Fig. 131 are obtained by producing BO in Fig. 128 and Fig. 129 respectively.

Let BO intersect the circle at the point E. Then in Fig. 130 we find $\angle AOB < \angle BOE$ and in Fig. 131 we find the angle under consideration $\angle AOB > \angle BOE$.

$\because$ $\angle BOE = 2$ right angles

$\therefore$ in Fig. 130, i.e., in Fig. 128, $\angle AOB$ is either an acute angle or an obtuse angle and in Fig. 131, i.e., in Fig. 129, $\angle AOB$ is a reflex angle.]

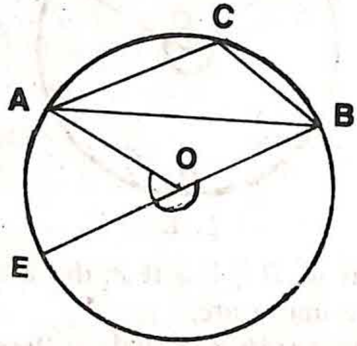


Fig. 131

Some Applications:

1. Prove that the circle drawn with hypotenuse of a right-angled triangle as diameter passes through the right angular vertex.

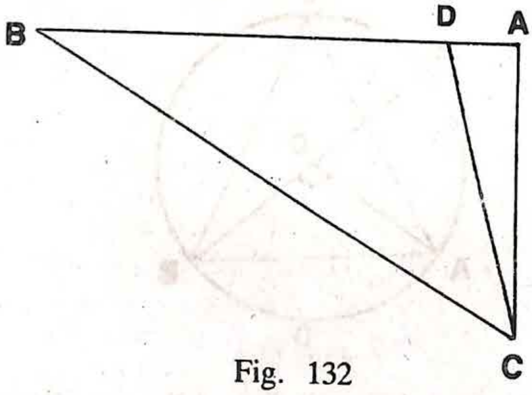


Fig. 132

Let ABC be a triangle right-angled at A.

To prove that the circle drawn with BC as diameter passes through the point A.

Proof: Let us suppose that the circle does not pass through the point A. Then, let the circle intersect BA at the point D.

$\therefore \angle BDC = 1$ right angle ($\because$ angle in a semicircle is a right angle).

$\angle BAC = 1$ right angle(by hypothesis)

$\therefore \angle BAC = \angle BDC$ (both being right angles),

Which is impossible if the points D and A do not coincide; since the exterior angle $\angle BDC$ of $\triangle ADC >$ the interior opposite angle $\angle BAC$.

$\therefore$ the circle passes through the point A.

[*Remarks:* (1) You have proved earlier that the median drawn through the right angular vertex of a right-angled triangle is half the hypotenuse. The property proved above can also be proved using this property. Try for yourself.

(2) It can also be proved with the help of Theorem 41. You try after learning the proof of that theorem.]

2. Prove that the circles drawn with two sides of a triangle, other than the greatest side, as diameters intersect at a point on the third side.

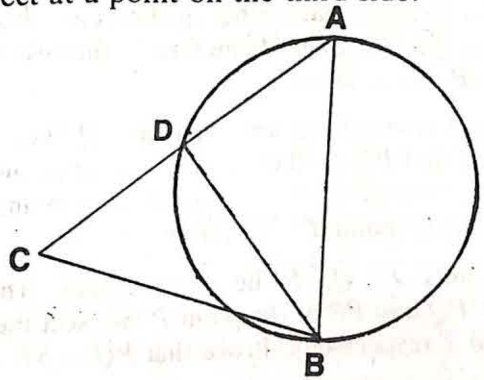


Fig. 133

Let ABC be a triangle and AC be the greatest side. Let us also suppose that the circle drawn with AB as diameter intersect AC at the point D.

To prove that the circle drawn with BC as diameter also passes through the point D.

Construction: D, B are joined.

Proof: $\because$ $\angle ADB$ is an angle in the semi-circle

$\therefore$ $\angle ADB = 1$ right angle

$\therefore$ $\angle CDB = 1$ right angle

$\therefore$ $\triangle CDB$ is a right-angled triangle. with BC as hypotenuse.

$\therefore$ the circle drawn with BC as diameter also passes through the point D.

[*Remarks:* (1) In case of an acute-angled triangle, this property will be valid for any two sides.

(2) It is clear that the foot of the perpendicular drawn to the third side from the opposite vertex is that point of intersection of two circles. It is obvious that the other point of intersection is the vertex, opposite to the third side.]

Exercise 19

1. Prove that the circle drawn with any one of the equal sides of an isosceles triangle as diameter bisects the unequal side.

2. Two circles intersect each other at the points P and Q. PA and PB are respectively the diameters of the two circles. Prove that A, Q, B are collinear.

3. R is the mid-point of the line segment PQ. Two circles are drawn with PR and PQ as diameters. Another straight line through the point P intersects the first circle at the point S and the second circle at the point T. Prove that $PS = ST$.

4. Three points P, Q, R lie on a circle. The two perpendiculars to PQ and PR at the point P intersect the circle at the point S and T respectively. Prove that $RQ = ST$.

5. $\triangle ABC$ is an acute-angled triangle. AP is the diameter of the circum-circle of $\triangle ABC$. BE and CF are perpendiculars to AC and AB respectively. They intersect each other at the point Q. Prove that $BPCQ$ is a parallelogram.

Theorem 40

The opposite angles of a cyclic quadrilateral are supplementary.

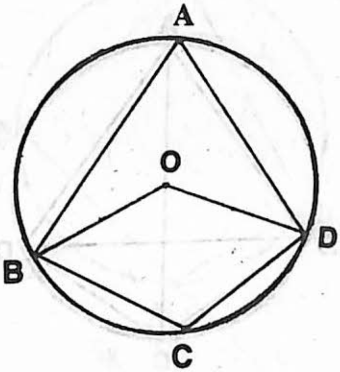


Fig. 134

Let *ABCD* be a cyclic quadrilateral in the circle, whose centre is *O*.
To prove that,

(i) $\angle BAD + \angle BCD = 2$ right angles,
(ii) $\angle ABC + \angle ADC = 2$ right angles,

Construction: B, O and *D*, *O* are joined.

Proof: $\angle BOD$ is the angle at the centre and $\angle BAD$ is the angle at the circumference standing on the same arc *BCD;*

$$\therefore \angle BOD = 2\angle BAD,$$

that is, $\angle BAD = \frac{1}{2} \angle BOD$

Again the reflex $\angle BOD$ is the angle at the centre and $\angle BCD$ is the angle at the circumference standing on the same arc *BAD;*

$\therefore \angle BCD = \frac{1}{2}$ the reflex $\angle BOD$
$\therefore \angle BAD + \angle BCD = \frac{1}{2}$ ($\angle BOD$ + reflex $\angle BOD$)
$= \frac{1}{2} \times 4$ right angles
$= 2$ right angles.

Similarly, by joining O, A and O, C; it can be proved that $\angle ABC + \angle ADC = 2$ right angles.

Alternative Proof:

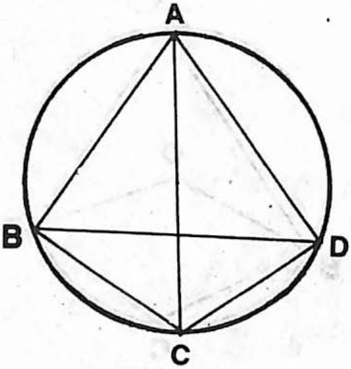


Fig. 135

Let ABCD be a cyclic quadrilateral in the circle, whose centre is O.

To prove that,

(i) $BAD + \angle BCD = 2$ right angles,

(ii) $\angle ABC + \angle ADC = 2$ right angles.

Construction: Two diagonals AC and BD are drawn.

Proof: $\because \angle BAC = \angle BDC$ (angle in the same segment of the circle)

and $\because \angle CAD = \angle CBD$ (angle in the same segment of the circle)

$\therefore \angle BAD = \angle BAC + \angle CAD = \angle BDC + \angle CBD$

$\therefore \angle BAD + \angle BCD = \angle BDC + \angle CBD + \angle BCD$

$= 2$ right angles.

[*Remarks:* (1) The alternative proof is given by Euclid.

(2) After proving that the sum of two opposite angles to be equal to 2 right angles, it can easily be shown that the sum of the other two opposite angles is equal to 2 right angles by using the property that the sum of four angles of a quadrilateral is equal to four right angles.]

Corollary: If any side of a cyclic quadrilateral be produced, the exterior angle so formed is equal to the interior opposite angle.

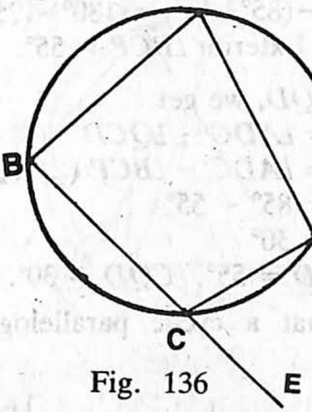


Fig. 136

In Fig. 136, the exterior $\angle DCE$ has been formed by producing the side BC to the point E, then, $\angle DCE$ = interior opposite $\angle BAD$.

$\because$ $ABCD$ is a cyclic qualdrilateral,

$\therefore$ $\angle BAD + \angle BCD = 180°$

Again, CD stands on the straight line BE,

$\therefore$ $\angle BCD + \angle DCE = 180°$,

$\therefore$ $\angle BCD + \angle DCE = \angle BAD + \angle BCD$.

$\therefore$ $\angle DCE = \angle BAD$.

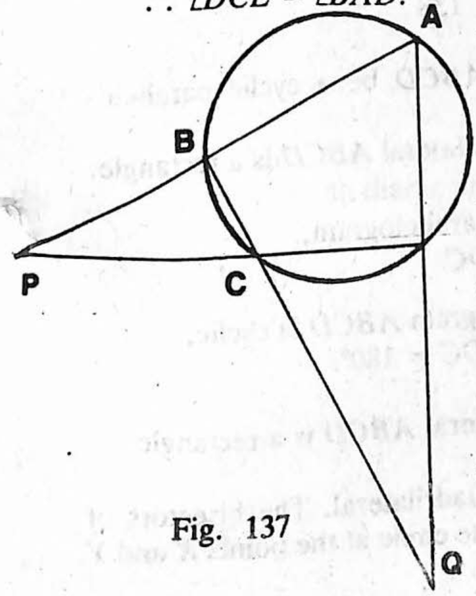


Fig. 137

Some Applications:

1. $ABCD$ is a cyclic quadrilateral. Sides AB and DC, when produced, meet at the point P and sides AD and BC, when produced meet at the point Q. If $\angle ADC = 85°$ and $\angle BPC = 40°$ then find the measures of $\angle BAD$ and $\angle CQD$.

The exterior $\angle PBC$ of

the cyclic quadrilateral $ABCD = \angle ADC = 85°$.

Again, in $\triangle BPC$, $\angle PBC = 85°$, $\angle BPC = 40°$.

$\therefore \angle BCP = 180° - (85° + 40°) = 180° - 125° = 55°$.

$\therefore \angle BAD =$ Exterior $\angle BCP = 55°$.

Again, from $\triangle CQD$, we get

$$
\begin{aligned}
\angle CQD &= \angle ADC - \angle QCD \\
&= \angle ADC - \angle BCP \ (\because \angle QCD = \angle BCP), \\
&= 85° - 55° \\
&= 30°
\end{aligned}
$$

$\therefore \angle BAD = 55°, \angle CQD = 30°$.

2. Prove that a cyclic parallelogram must be a rectangle.

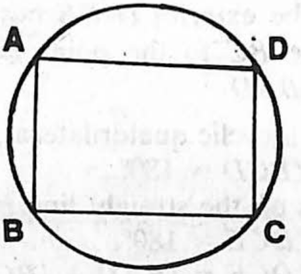


Fig. 138

Let the quadrilateral $ABCD$ be a cyclic parallelogram.

To prove that the quadrilateral $ABCD$ is a rectangle.

Proof: $\because$ $ABCD$ is a parallelogram,

$\therefore \angle ABC = \angle ADC$.

Again, $\because$ the parallelogram $ABCD$ is cyclic,

$\therefore \angle ABC + \angle ADC = 180°$.

$\therefore \angle ABC = 90°$.

$\therefore$ the quadrilateral $ABCD$ is a rectangle.

3. $ABCD$ is a cyclic quadrilateral. The bisectors of $\angle DAB$ and $\angle BCD$ intersect the circle at the points X and Y

respectively. Prove that XY is a diameter of that circle.

Construction: A, Y are joined.

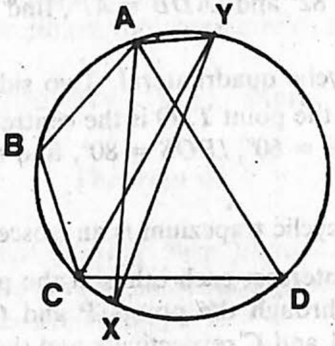


Fig. 139

Proof: $\angle YAD$ and $\angle YCD$ stand on the same arc;

$\therefore \angle YAD = \angle YCD = \frac{1}{2}\angle BCD$ ($\because$ CY is the bisector of $\angle BCD$)

$\therefore \angle XAY = \angle XAD + \angle YAD$

$= \frac{1}{2}\cdot\angle BAD + \frac{1}{2}\angle BCD$ ($\because$ AX is the bisector of $\angle DAB$)

$= \frac{1}{2}(\angle BAD + \angle BCD)$

$= \frac{1}{2} \times 180°$ ($\because$ $ABCD$ is a cyclic quadrilateral)

$= 90°$

$\therefore$ $XCBAY$ is a semi-circle

$\therefore$ XY is the diameter of the circle.

Exercise 20

1. side AB of cyclic quadrilateral $ABCD$ is produced to the point X. If $\angle XBC = 82°$ and $\angle ADB = 47°$, find the measure of $\angle BAC$.

2. $PQRS$ is a cyclic quadrilateral. Two sides PQ, SR are produced to meet at the point T. O is the centre of the circle. If $\angle POQ = 110°$, $\angle QOR = 60°$, $\angle ROS = 80°$, find the measures of $\angle RQS$ and $\angle QTR$.

3. Prove that a cyclic trapezium is an isosceles trapezium.

4. Two circles intersect each other at the points P and Q. Two straight lines through the points P and Q intersect one circle at the points A and C respectively and the other circle at the points B and D respectively. Prove that $AC \parallel BD$.

5. $ABCD$ is a cyclic quadrilateral. The side BC is produced to the point E. Prove that the bisectors of $\angle BAD$ and $\angle DCE$ meet on the circle.

6. $ABCD$ is a cyclic quadrilateral. The chord DE is the external bisector of $\angle BDC$. Prove that AE (or AE produced) is the external bisector of $\angle BAC$.

7. Two straight lines are drawn through any point X, exterior to a circle. One of them intersects the circle at the points A, B respectively and the other at the points C, D respectively. Prove that $\triangle XAC$ and $\triangle XBD$ are equiangular.

8. Two circles intersect each other at the points G and H. A striaght line through the point G intersects two circles at the points P and Q, and the straight line through the point H parallel to PQ intersects the two circles at the points R and S. Prove that $PQ = RS$.

9. In $\triangle ABC$, $AB = AC$, E is any point on BC produced. The circumcircle of $\triangle ABC$ intersects AE at the point D. Prove that $\angle ACD = \angle AEC$.

4.6 *Conditions for four points to be concyclic:*

You have already known that the vertices of an isosceles trapezium lie on the same circle, that is, they are concyclic.

We will now prove two more criteria.

Theorem 41

If a line segment joining two points subtends equal angles at two points on the same side of the line segment, then the four points are concyclic.

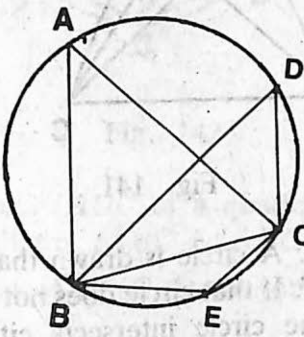


Fig. 140

Let the line segment joining the points B and C subtend two equal angles at the points A and D on the same side of the line segment; that is, $\angle BAC = \angle BDC$.

To prove that A, B, C, D are concyclic.

Construction: The circle passing through the points A, B, C is drawn. Through the point C, a straight line parallel to BD is drawn which intersects the circle at the point

B, E are joined.

Proof: $\because$ $ABEC$ is cyclic quadrilateral,

$\therefore \angle BAC + \angle BEC = 180°$.

But, $\angle BAC = \angle BDC$ (by hypothesis),

$\therefore \angle BDC + \angle BEC = 180°$.

Again, $CE \parallel DB$ and BE is the transversal,

$\therefore \angle DBE + \angle BEC = 180°$

$\therefore \angle BDC + \angle BEC = \angle DBE + \angle BEC$

$\therefore \angle BDC = \angle DBE$

$\therefore$ *BDCE* is an isosceles trapezium

Again, *A, B, E, C* are concyclic (by construction),

$\therefore$ *A, B, C, D* are concyclic.

Alternative Proof :

To prove that *A, B, C, D* are concyclic.

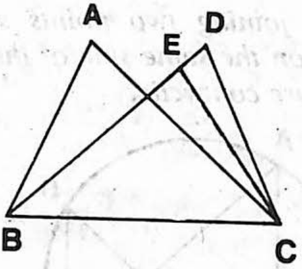


Fig. 141

Construction : A circle is drawn that passes through the points *A, B, C*. If that circle does not pass through the point *D*, then the circle intersects either *BD* or *BD* produced. Let us suppose that in this case, the circle intersects *BD* at the point *E*. *C, E* are joined.

Proof : $\angle ABC$ and $\angle BEC$ are angles at the same segment of the circle.

$\therefore \angle BAC = \angle BEC$

But $\angle BAC = \angle BDC$ (by hypothesis)

$\therefore \angle BEC = \angle BDC$.

But this is impossible if the points *D* and *E* do not coincide; for $\angle BEC$, the exterior angle of $\triangle DEC > \angle BDC$

$\therefore$ the point *D* lies on the circle,

$\therefore$ *A, B, C, D* are concyclic.

Theorem 42

If two opposite angles of a quadrilateral are supplementary, then the vertices of the quadrilateral will be concyclic.

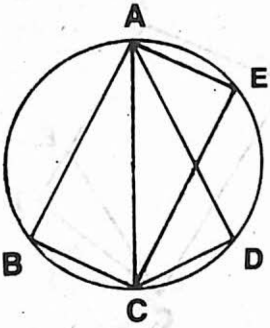


Fig. 142

Let $\angle ABC$ and $\angle ADC$ of a quadrilateral $ABCD$ be supplementary; that is, $\angle ABC + \angle ADC = 180°$.

To prove that, A, B, C, D are concyclic.

Construction : A circle is drawn through the points A, B, C. A, C are joined. Let us suppose E to be such a point on that circle that lies on that side of AC in which the point D lies. A, E, and E, C are joined.

Proof : $\because$ $ABCE$ is a cyclic quadrilateral,

$\therefore \angle ABC + \angle AEC = 180°$.

But $\angle ABC + \angle ADC = 180°$ (by hypothesis),

$\therefore \angle ABC + \angle AEC = \angle ABC + \angle ADC$

$\therefore \angle AEC = \angle ADC$.

$\because$ $\angle AEC$ and $\angle ADC$ are two equal angles on the same side of AC,

$\therefore$ A, C, D, E are concyclic.

But $\because$ A, B, C, E are concyclic,

$\therefore$ A, B, C, D are concyclic.

Alternative Proof :

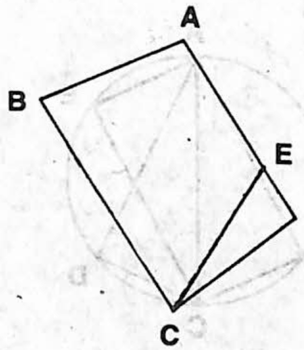


Fig. 143

Let $\angle ABC$ and $\angle ADC$ of a quadrilateral $ABCD$ be supplementary; that is,
$\angle ABC + \angle ADC = 180°$.

To prove that, A, B, C, D are concyclic.

Construction: A circle is drawn through the points A, B, C. If that circle does not pass through the point D, then the circle intersects either AD or AD produced. Let us suppose that in this case, the circle intersects AD at the point E. C, E are joined.

Proof : $\because$ $ABCE$ is a cyclic quadrilateral,

$\therefore$ $\angle ABC + \angle AEC = 180°$.

But $\angle ABC + \angle ADC = 180°$ (by hypothesis),

$\therefore$ $\angle ABC + \angle AEC = \angle ABC + \angle ADC$,

$\therefore$ $\angle AEC = \angle ADC$.

But this is impossible if the points D and E do not coincide; for $\angle AEC$, the exterior angle of $\triangle DEC > \angle ADC$.

the point D lies on the circle,

$\therefore$ A, B, C, D are concyclic.

Some Applications :

1. In $\triangle ABC$, the perpendicular drawn to BC through the point A intersects BC at the point D and the perpendicular drawn to CA through the point B intersects CA at the point E. The two perpendiculars intersect each other at the point O. Prove that *(i)* A, B, D, E are concyclic, *(ii)* C, D, O, E are concyclic.

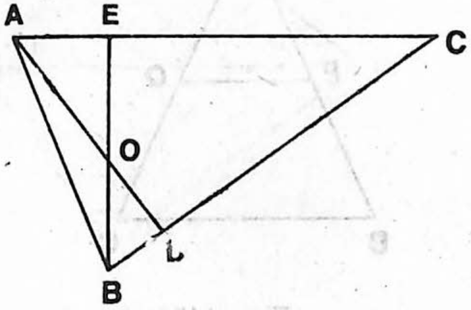


Fig. 144

Proof : *(i)* $\angle AEB$ = 1 right angle ($\because$ $BE \perp CA$),

$\angle ADB$ = 1 right angle ($\because$ $AD \perp BC$),

$\angle AEB = \angle ADB$.

But they are two angles lying on the same side of AB,

$\therefore$ A, B, D, E are concyclic.

(ii) $\angle OEC$ = 1 right angle ($\because$ $BE \perp CA$),

$\angle ODC$ = 1 right angle ($\because$ $AD \perp BC$),

$\angle OEC + \angle ODC$ = 2 right angles.

But they are two opposite angles of a quadrilateral $CDOE$,

$\therefore$ C, D, O, E are concyclic.

2. In an isosceles triangle ABC, $AB=AC$. A straight line parallel to BC intersects AB and AC at the points P and Q respectively. Prove that, $BPQC$ is a cyclic quadrilateral.

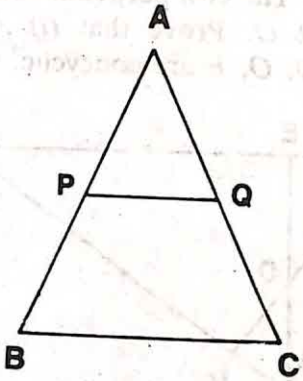


Fig. 145

Proof : $\because$ in the isosceles triangle ABC, $AB=AC$,

$\therefore \angle ACB = \angle ABC$.

Again, $\because PQ \parallel BC$,

$\because \angle ABC =$ corresponding $\angle APQ$.

$\angle APQ = \angle ACB$.

but $\angle BPQ + \angle APQ = 2$ right angles,

$\therefore \angle BPQ + \angle ACB = 2$ right angles,
that is, $\angle BPQ + \angle QCB = 2$ right angles,

But they are two opposite angles of the quadrilateral $BPQC$.

$\therefore BPQC$ is a cyclic quadrilateral.

Exercise 21

1. In Fig 146, $\angle BCD=88°$, $\angle ABD=48°$, $\angle ADB=40°$. Prove that A, B, C, D are concyclic.

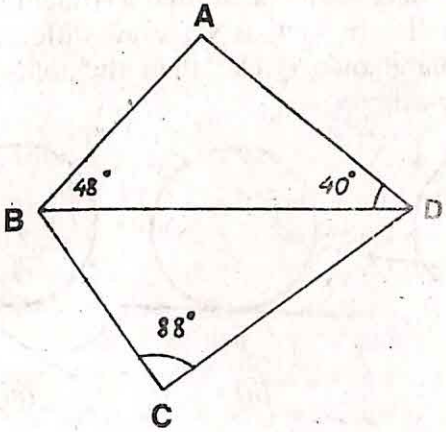


Fig. 146

2. $\triangle ABC$ is right-angled at B. DE is perpendicular to AC from any point D on AB. Prove that B, D, E, C are concyclic.

3. A semi-circle is drawn with AB as diameter. Two chords AP and BQ intersect each other at the point R. Perpendicular to AB from R meets AB at the point S. Prove that *(i)* B, S, R, P are concylic, *(ii)* A, S, R, Q are concyclic.

4. R and S are the mid-points of two chords AP and AQ of a circle with centre O. Prove that O, R, A, S are concyclic.

5. BE and CF are respectively perpendiculars to sides AC and AB of the triangle ABC. Prove that B, C, E, F are concyclic. Hence, prove that $\triangle AEF$ and $\triangle ABC$ are equiangular.

6. $ABCD$ is a parallelogram. A circle passing through the points A and B intersects AD and BC (or AD and BC produced) at the points E and F respectively. Prove that E, F, C, D (or E, C, F, D) are concyclic.

7. $ABCD$ is a quadrilateral, so that $AB+DC=AD+BC$. The points P, Q, R, S are taken on AB, BC, CD, DA respectively in such a way that $AP=AS$, $BP=BQ$ and $CQ=CR$. Prove that $PQRS$ is a cyclic quadrilateral.

12

4·7 Tangent :

We have already seen that if a straight line intersects a circle, then it intersects the circle at most at two points. But it does not necessarily mean that a straight line always intersects a circle. In fact, if we draw different straight lines in the plane of a circle, then the following three situations may arise.

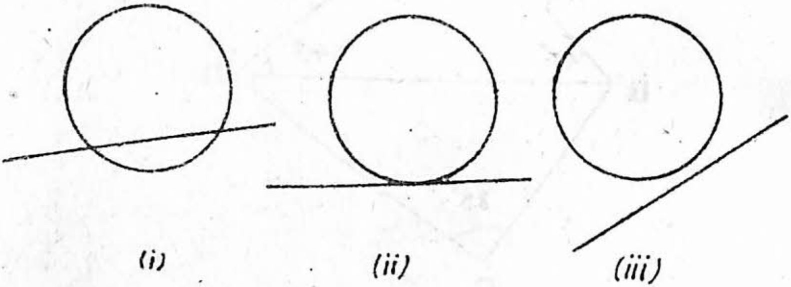


Fig. 147

(i) Straight line intersects the circle at two points [Fig. 147*(i)*]. In this case, we call the straight line a *secant* of the circle.

(ii) Straight line will touch the circle; that is the straight line meets the circle at one and only one point. [Fig. 147*(ii)*]. In this case, we call the straight line a *tangent* to the circle.

(iii) Straight line does not intersect the circle, that is they have no common point [Fig. 147*(iii)*].

From the above discussion we can now define a secant and a tangent.

Definition : 1. *Secant*—The straight line, which intersects a circle at two points, is called the *secant* of the circle.

2. Tangent—The straight line, which meet a circle at one and only one point, is called the *tangent* to the circle at that point. The point where the circle and the tangent meet, that is, the point where the tangent touches the

circle is called the *point of contact.*

[*Remark :* Of all the points on the tangent only the point of contact will lie on the circle, the other points will be exterior to the circle.]

Now let us further analyse the relation between a circle and a straight line. Let a straight line intersect a circle with centre O at two points A and B. Now if the straight line is rotated keeping the point A fixed (for example, in Fig. 148.it has been rotated anti-clockwise) then we see that the other point of intersection gradually comes nearear and nearer to the point A.

As a result of this rotation the position of the straight line may be such that the other point of intersection coincides with the point A. Then we may say that a straight line intersecting a circle at two coincident points is a tangent to the circle at that point.

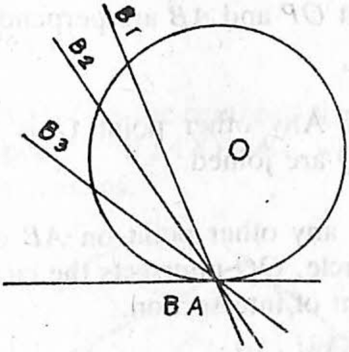


Fig. 148

4.8 Now we will establish, the relation between the tangent to a circle and the radius through the point of contact in the next theorem.

Theorem 43

The tangent to a circle at any point on it is perpendicular to the radius through the point of contact.

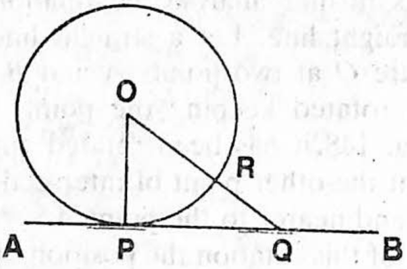


Fig. 149

Let AB be the tangent to the circle with centre O at the point P and OP be the radius through the point P.

To prove that OP and AB are perpendicular to each other.

Construction: Any other point Q is taken on the tangent AB. O, Q are joined.

Proof: Since any other point on AB excepting P is exterior to the circle, OQ intersects the circle at a point. Let R be the point of intersection.

$\therefore OR<OQ$ [$\because$ R is a point between O and Q.]

Again $OR=OP$ (radii of the same circle).

$\therefore OP<OQ$.

$\because$ the point Q is any point on AB,

$\therefore OP$ is the least of all the line segments drawn from the point O to AB.

$\therefore OP \perp AB$.

Corollary: 1. At any point on a circle, the perpendicular drawn to the radius through that point is the tangent to that circle.

2. At a given point on a circle one and only one tangent can be drawn.

3. The perpendicular drawn to any tangent to a circle at the point of contact passes through the centre of the circle.

4. The perpendicular drawn to a tangent to a circle from the centre of the circle passes through the point of contact.

[*Remark:* Only one perpendicular can be drawn to a straight line at a point on it or only one perpendicular can be drawn to a straight line from a point outside it. From these two properties we can arrive at the above corollaries.]

Some Applications :

1. In Fig. 150, O is the centre of the circle. YX is the tangent at the point Y. If $\angle AXY=40°$, what is the measure of $\angle AYO$? Give reasons.

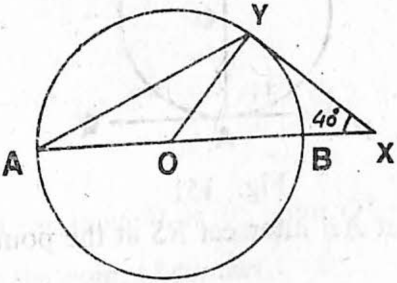


Fig. 150

Answer : $\angle AYO=25°$.

Proof: $\because$ XY is the tangent and OY is the radius through the point of contact,

$\therefore \angle XYO = 90°$,

$\therefore$ in $\triangle OXY$,

$$\angle XOY = 180° - (90° + 40°)$$
$$= 180° - 130°$$
$$= 50°.$$

Again, $OA = OY$ (radii of same circle),

$\therefore$ in $\triangle AOY$, $\angle OAY = \angle OYA$

$\therefore$ Exterior $\angle XOY = \angle OAY + \angle OYA$
$= 2\angle OYA$

$\therefore 2\angle OYA = 50°$,
or, $\angle OYA = 25°$,
$\therefore \angle AYO = 25°$.

2. AB is a diameter of a circle with centre O. PAQ is the tangent to the circle at the point A. RS is a chord parallel to the tangent PAQ. Prove that AB is perpendicular bisector of RS.

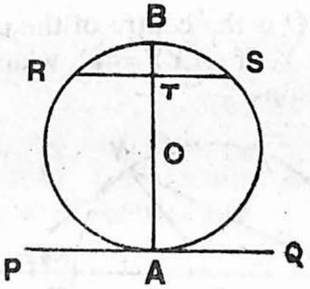


Fig. 151

Proof: Let AB intersect RS at the point T.

$\because$ PAQ is the tangent at the point A and AB is a diameter,

$\therefore AB \perp PAQ$.

Again, $\because PAQ \parallel RS$ and AB is the transversal; and $AB \perp PAQ$ (proved),

$\therefore AB \perp RS$; that is $OT \perp RS$.

$\because OT \perp RS$,

$\therefore$ T is the mid-point of RS.

$\therefore AB$ is the perpendicular bisector of RS.

3. PQ is a diameter of a circle with centre O. APB and CQD are the two tangents to the circle at the points P and Q. Prove that $AB \parallel CD$.

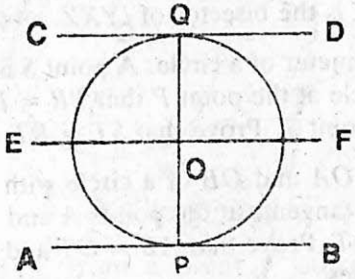


Fig. 152

Construction: A straight line EF through the point O parallel to AB is drawn.

Proof: $OP \perp AB$ ($\because APB$ is the tangent and OP is the radius through the point of contact).

Again, $\because AB \parallel EF$,

$\therefore OP \perp EF$

$\therefore OQ \perp EF$ ($\because OP$, OQ are the same straight line)

$\therefore CQD$ is the tangent at the point Q and OQ is the redius through the point of contact

$\therefore OQ \perp CD$

again, $EF \parallel CD$ ($\because OQ \perp EF$ and $OQ \perp CD$)

$\therefore AB \parallel CD$.

Exercise 22

1. AB is a chord of the circle with centre O. The tangent at the point B intersects AO produced at the point T. If $\angle BAT = 21^\circ$, find the measure of $\angle BTA$.

2. XY is a diameter of a circle. Another point A lies on the circle. PAQ is tangent to the circle at the point A. The perpendicular drawn to the tangent intersects PAQ at the point Z. Prove that XY is the bisector of $\angle YXZ$.

3. PR is a diameter of a circle. A point S is so taken on the tangent to the circle at the point P that $PR = PS$. RS intersects the circle at the point T. Prove that $ST = RT = PT$.

4. Two radii OA and OB of a circle with centre O are at right angles. Two tangents at the points A and B intersect each other at the point T. Prove that $AB = OT$ and they bisect each other at right angles.

5. P is any point on a diameter of a circle with centre O. The perpendicular drawn to the diameter at the point O intersects that circle at the point Q. QP produced intersects OP at the point S. Prove that $SP = SR$.

6. X is any point on the tangent to the circle with centre O at the point A. A secant through the point X intersects the circle at the points Y and Z. P is the mid-point of YZ. Prove that $XAPO$ or $XAOP$ is a cyclic quadrilateral.

7. Two chords AC and BD intersect each other at the point O. Two tangents at the points A and B intersect each other at the point P and two tangents at the points C and D intersect each other at the point Q. Prove that $\angle P + \angle Q = 2\angle BOC$.

8. QR is a chord of a circle with centre O. Two tangent at the points Q and R intersect each other at the point P. QM is a diameter of the circle. Prove that $\angle QPR = 2\angle RQM$.

4·9 We have seen that two tangents drawn at the end points of a diameter are parallel to each other. If two tangents are drawn at other two points, then they intersect each other at a point outside the circle; that is we may say that two tangents can be drawn to a circle from a point outside it.

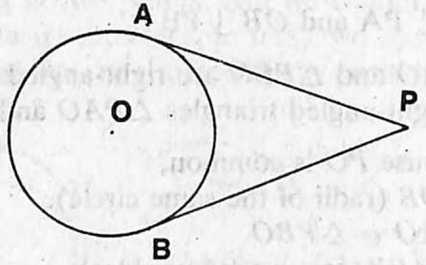


Fig. 153

In Fig.153, from a point P outside the circle with centre O, two tangents PA and PB can be drawn.

We establish the relation between these two tangents in the next theorem.

Theorem 44

If two tangents be drawn to a circle from a point outside it, then the line segments joining the points of contact and the exterior point are equal and they subtend equal angles at the centre.

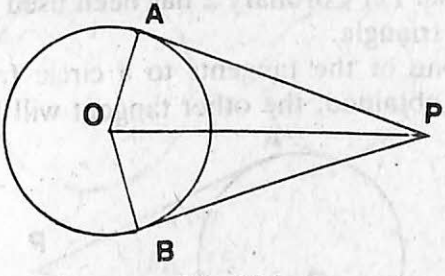


Fig. 154

Let P be a point outside the circle with centre O. From the point P two tangents PA and PB are drawn, whose points of contact are A and B respectively. O, A; O,

B; O, P are joined. As a result PA and PB subtend $\angle POA$ and $\angle POB$ respectively at the centre O.

To prove that (i) $PA = PB$ (ii) $\angle POA = \angle POB$

Proof: PA and PB are tangents and OA and OB are the radii through the points of contact.

$\therefore$ $OA \perp PA$ and $OB \perp PB$

$\therefore$ $\triangle PAO$ and $\triangle PBO$ are right-angled triangles.
In the right-angled triangles $\triangle PAO$ and $\triangle PBO$,

Hypotenuse PO is common,
$OA = OB$ (radii of the same circle).
$\therefore$ $\triangle PAO \cong \triangle PBO$
$\therefore$ $PA = PB$ (corresponding sides) . . (i)

and $\angle POA = \angle POB$ (corresponding angles) . . (ii)

Corollary: 1. The line joining a point outside a circle and the centre bisects the angle included by two tangents drawn from that exterior point.

2. The bisector of the angle included by two tangents drawn to a circle from a point outside it passes through the centre.

3. If two tangents drawn to a circle at two points on it intersect each other, then the lengths of the line segments from the point of intersection to the points of contact are equal.

[*Remarks:* (1) Corollary 2 has been used to draw the incircle of a triangle.

(2) If one of the tangents to a circle from a point outside it is obtained, the other tangent will be obtained easily.

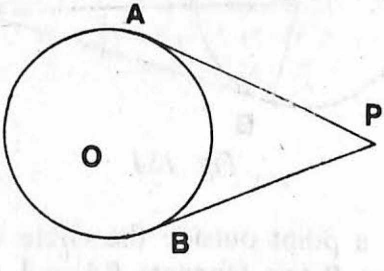
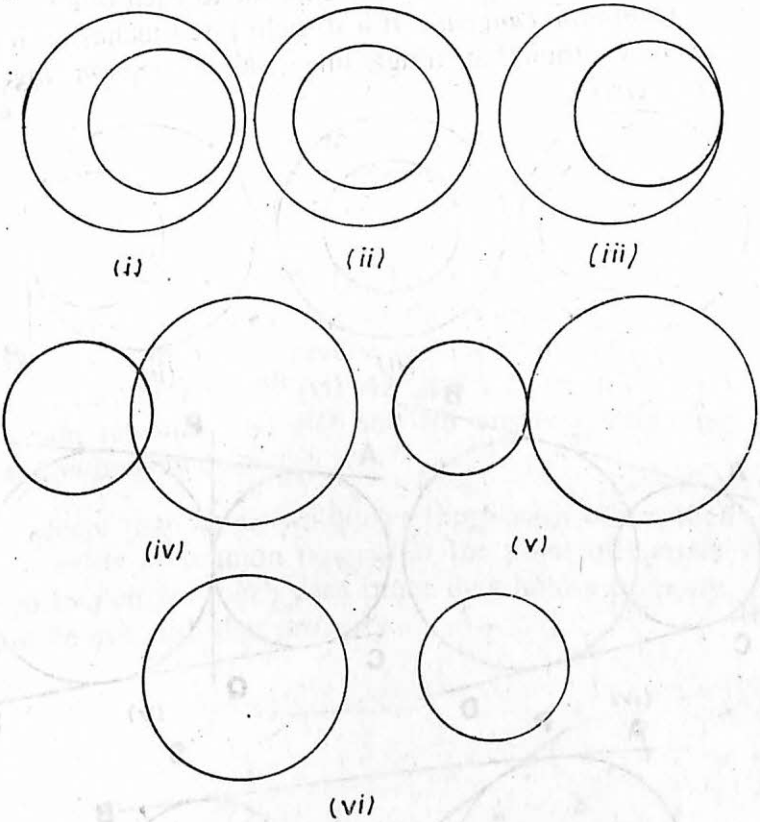


Fig. 155

Let the tangent PA to a circle be drawn. Now draw an arc of a circle with centre P and radius equal to PA. The arc intersects the circle at the point B. Then PB will be the second tangent.]

4·10 We have seen that two circles may intersect each other at most at two points. But we cannot say that two circles will always intersect. In fact, two circles drawn on the same plane may be situated in any position as shown in figure 156 :



(i) (ii) (iii)

(iv) (v)

(vi)

Fig. 156

(i) One of the two circles may be in the interior of the other circle and they do not meet at any point.

(ii) Two circles may be concentric. In that case also, they do not meet at any point.

(iii) One of the two circles may be in the interior of the other circle and they meet at one and only one point. In this case we say that two circles touch each other internally.

(iv) Two circles intersect each other at two points.

(v) One of the two circles be exterior to the other circle, but they meet at one and only one point. In this case, we say that two circles touch each other externally.

(vi) Neither two circles intersect nor they touch each other, although they are exterior to each other.

Common Tangent : If a straight line touches each of two circles, then that straight line is called *common tangent to two circles.*

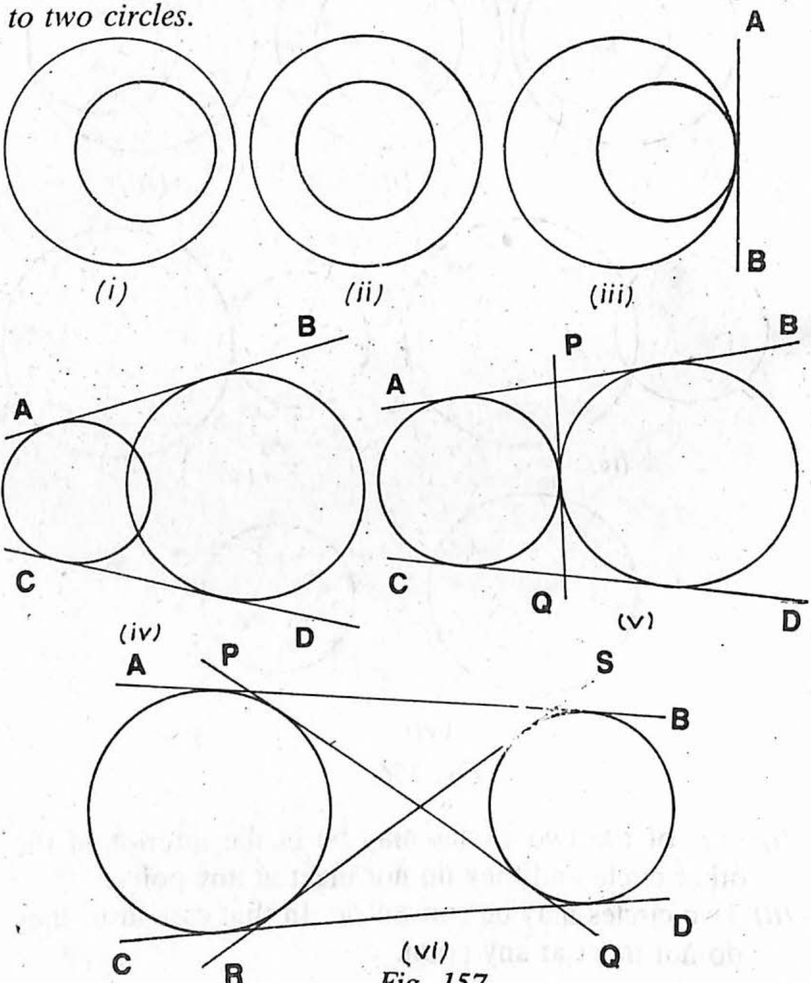


Fig. 157

If we consider the different positions of two circles, then we find that there is no common tangent in first and second cases, there is one common tangent in the third case, two common tangents in fourth case, three common tangents in fifth case and four common tangents in sixth case.

Of those common tangents some are so situated that two circles lie on their same side and some others are so situated that two circles lie on their opposite sides. The common tangents in the first case are called *direct common tangents* and the common tangents in the second case are called *transverse common tangents*.

(i) In case of fig. 157 *(iii)* we call the common tangent AB as direct common tangent.

(ii) In case of fig. 157 (iv) AB and CD are two direct common tangents.

(iii) In case of fig. 157 (v) AB and CD are two direct common tangents, but the tangent PQ may be said to be a transverse common tangent.

(iv) In case of fig. 157 (vi) AB and CD are two direct common tangents and PQ and RS are two transverse common tangents.

We accept that when two circles touch each other, then there exists a common tangent at the point of contact. when two circles touch each other they hold a property. Now we establish that property.

Theorem 45

If two circles touch each other, then the point of contact will lie on the line joining the two centres.

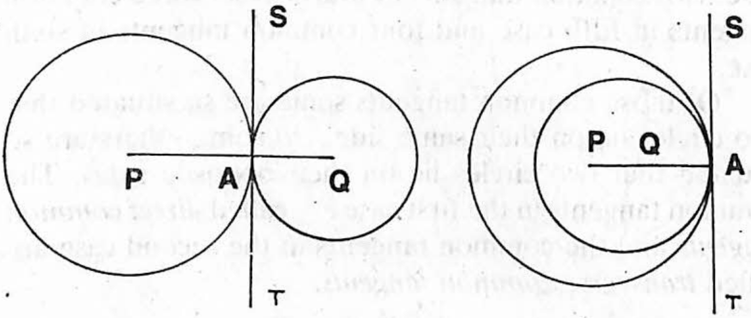


Fig. 158

Let two circles with centres P and Q touch each other at the point A.

To prove that P, A, Q are collinear.

Construction : P, A and Q, A are joined.

Proof : Since two circles touch each other at the point A, there exits a common tangent at the point A. Let ST be the common tangent, which touches both the circles at the point A.

$\because$ ST is the tangent to the circle with centre P and PA is the radius through the point of contact,

$\therefore PA \perp ST$

Again, $\because ST$ is the tangent to the circle with centre Q and QA is the radius through the point of contact,

$\therefore QA \perp ST$

$\therefore PA$ and QA are both perpendicular to ST at the same point.

$\therefore PA$ and QA lie on the same straight line; that is, the three points, P, A, Q are collinear.

Corallary : 1. If two circles touch each other, the straight line through centre of one circle and the point of contact passes through the centre of other circle.

2. If two circles touch each other externally, the distance between the centres of two circles is equal to sum of the lengths of the radii of the circles. In Fig. 159 $PQ=PA+AQ$.

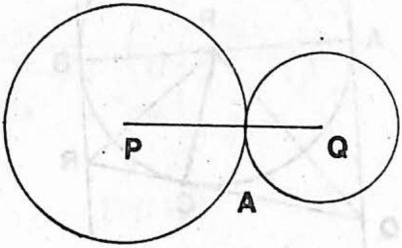


Fig. 159

3. If two circles touch each other internally, the distance between the centres of two circles is equal to the difference of the lenghts of the radii of two circles. In Fig. 160 $PQ=PA-AQ$.

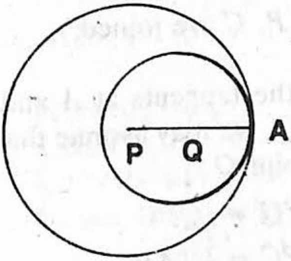


Fig. 160

Some Applications :

1. P is the centre of a circle. A tangent to the circle at a point on it intersects two tangents at the end points of a

diameter at Q and R respectively. Prove that $\angle QPR$ is a right angle.

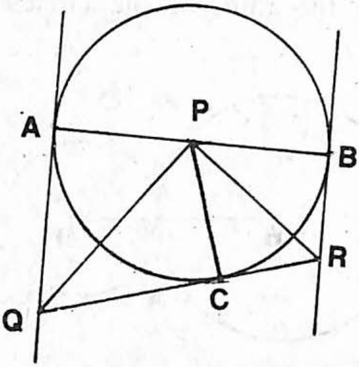


Fig. 161

Let AB be a diameter of a circle with centre P and C be a point on the circle. The tangent at the point C intersects the tangents at the points Q and R respectively. Q, P and R, P are joined.

To prove that $\angle QPR = 1$ right angle.

Construction: P, C are joined.

Proof: Since the tangents at A and C intersect each other at the point Q, we may assume that two tangents are drawn from the point Q.

$$\therefore \angle APQ = \angle QPC$$

$$\therefore \angle QPC = \tfrac{1}{2} \angle APC$$

Similarly, $\angle CPR = \tfrac{1}{2} \angle CPB$

$$\therefore \angle QPR = \angle QPC + \angle CPR$$

$$= \tfrac{1}{2} (\angle APC + \angle CPB)$$

$$= \tfrac{1}{2} \angle APB$$

$$= \tfrac{1}{2} \times 180^\circ$$

$$= 90^\circ$$

2. AB and AC are the tangents to the circle with centre O. Prove that AO bisects the chord of contact at right angles.

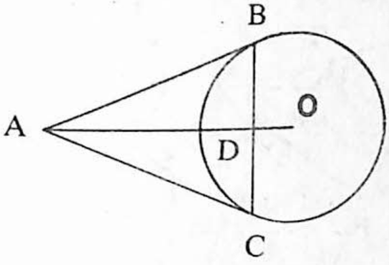


Fig. 162

Let AO intersect the chord of contact BC at the point D.

To prove that AO is the perpendicular bisector of BC.

Proof: $\because$ AB and AC are the tangents to the circle with centre O, $\quad\therefore AB=AC$.

and AO is the bisector of $\angle BAC$;

that is, $\angle BAD = \angle CAD$.

In $\triangle ABD$ and $\triangle ACD$, $AB=AC$ (proved),

AD is common,

and included $\angle BAD$ = included $\angle CAD$

$\therefore \triangle ABD \cong \triangle ACD$.

$\therefore BD = DC$ (corresponding sides).

and $\angle BDA = \angle CDA$ (corresponding angles)

$\therefore AD \perp$ BC

$\therefore$ AO is the perpendicular bisector of BC.

3. Two circles with centres P and Q touch each other at the point A. A direct common tangent to two circles touch them at the points R and S respectively. Prove that *(i)* the common tangent at the point A bisects the line segment RS at the point T, *(ii)* $\angle RAS$

= 1 right angle, *(iii)* If PT and QT intersect AR and AS at the points C and B respectively, then $ABTC$ will be a rectangle.

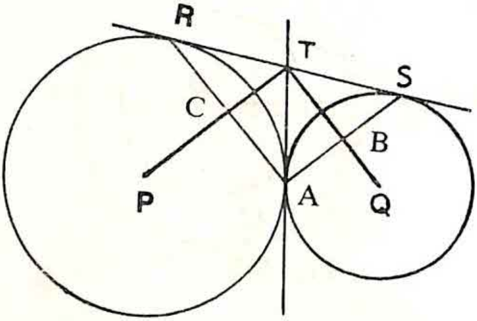


Fig. 163

Proof: *(i)* Let the tangent at the point A intersect RS at the point T.

$\therefore$ TR and TA are two tangents to the circle with centre P from the point T.

$$TR = TA$$

Similarly, $TS = TA$,

$$\therefore TR = TS$$

$\therefore$ AT bisects RS,

that is, the common tangent at A bisects RS.

(ii) In $\triangle ATR$,

$$TR = TA$$

$$\therefore \angle TAR = \angle TRA$$

Similarly, $\angle TAS = \angle TSA$.

$$\therefore \angle RAS = \angle TAR + \angle TAS$$

$$= \angle TRA + \angle TSA$$

$$\therefore \angle RAS = 90^\circ$$

$$= 1 \text{ right angle.}$$

(iii) PT is the bisector of $\angle RTA$ and QT is the bisector of $\angle ATS$.

$$\therefore PT \perp QT$$

$$\therefore \angle PTQ = 1 \text{ right angle.}$$

Again, $PT \perp RA$ and $QT \perp SA$,

$\therefore \angle ACT = \angle ABT = 1$ right angle.

$\therefore ABTC$ is a rectangle.

4. If a quadrilateral is circumscribed about a circle, prove that the sum of any two opposite sides of the quadrilateral is equal to the sum of the other two opposite sides.

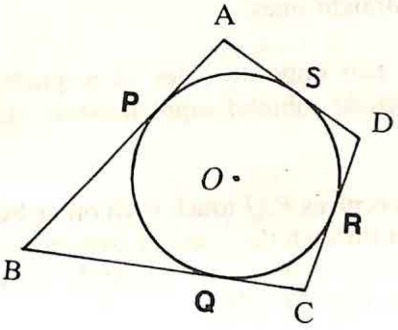


Fig. 164

Let the quadrilateral $ABCD$ be circumscribed about the circle with centre O. Sides AB, BC, CD, DA touch the circle at the points P, Q, R, S respectively.

To prove that $AB + DC = AD + BC$.

Proof: $\because$ AP and AS are tangents to the circle,

$\therefore AP = AS$.

Similarly, $BP = BQ$, $DR = DS$ and $CR = CQ$.

$$\begin{aligned}\therefore AB + DC &= AP + BP + DR + CR \\ &= AS + BQ + DS + CQ \\ &= AS + DS + BQ + CQ \\ &= AD + BC.\end{aligned}$$

Exercise 23

1. Two circles touch externally at the point A. PQ is a direct common tangent to those two circles. Prove that $\angle PAQ$ is a right angle. If PR be a diameter, prove that Q, A, R are collinear.

2. Prove that the centre of the circle which is touched by two intersecting straight lines lies on the bisector of the angle included by those two straight lines.

3. Prove that any two opposite sides of a quadrilateral circumscribed about a circle subtend supplementary angles at the centre.

4. Two circles with centres P, Q touch each other externally. A straight line drawn through the point of contact intersects the circle with centre P at the point A and the circle with centre Q at the point B. Prove that $AP \parallel BQ$.

. . Two circles with centres X and Y touch each other externally. A straight line drawn through the point of contact intersects the circle with centre X at the point R and the circle with centre Y at the point S. Prove that the tangent to the circle with centre X at the point R is parallel to the tangent to the circle with centre Y at the point S.

6. Three equal circles touch each other externally. Prove that the centres of those circles are the vertices of an equilateral triangle.

7. From a point A outside the circle with centre O, tangents AP and AQ are drawn. Their points of contact are P and Q respectively. Prove that A, P, O, Q, are concyclic.

8. From a point A outside a circle, two tangents AB and AC are drawn. The tangent at the point X on the minor are BC intersects AB and AC at the points D and E respectively. Prove that the perimeter of $\triangle 1DE = 2AB$.

9. P and Q are two points outside a circle. If the angle

included by the tangents drawn from the point P be equal to the angle included by the tangents drawn from the point Q, then prove that the distances of P and Q from the centre are equal.

10. AB and AC are two fixed tangents to a circle with centre O. Another tangent intersects the tangent AB at the point P and the tangent AC at the point Q. Prove that for any position of the tangent the measure of $\angle POQ$ remains unchanged.

CHAPTER 5

Similarity

5.1. Ratio and Proportion :

You are familliar with the ideas of 'Ratio' and 'Proportion' in arithmetic and algebra. Can you apply these ideas in geometry? Let us see.

Suppose, your school is 1 km towards east from your house and the house of one of your friends is 400 metres towards north. How will you show these distances in your exercise book? Let *A* be the position of your house and you denote the position of your school by a point *B*, at a distance of 5 cm towards east from the point *A*. Now, where will you locate the position of your friend's house? You are to locate it by a point *C* towards north of *A* and at a distance of 2 cm from *A*. For,

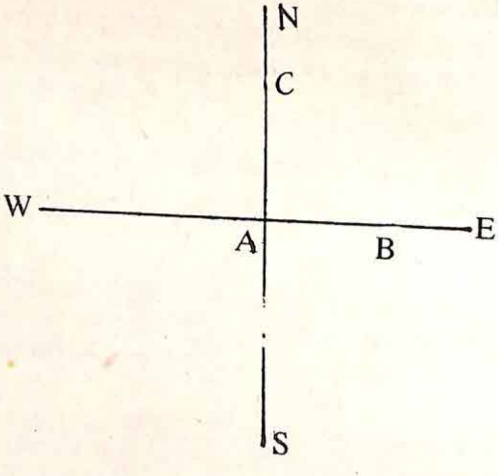


Fig. 165

1 km : 400 metres = 1000 metres : 400 metres = 5 : 2
Again, 5 cm : 2 cm = 5 : 2;

that is, the lengths of the line segments AB and AC will bear a ratio 5 : 2.

Let us take another example. You have seen wall map of India in your geography class. You also know how to draw an outline of the map of India or how to locate in an outline of the map the positions of the places situated in India.

The distances of Delhi and Bombay from Calcutta are approximately 1287 km and 1529 km respectivety. These distances are shown in a wall map by 31 cm and 37 cm respectively. How will you show these distances in the outline of the map? Your outline of the map is surely much smaller than the wall map. So, although the two maps are of same shape, the distances will not be same. If the wall map be 5 times the outline of the map, you will have to show the distances by 6·2 cm and 7·4 cm respectively; for, 31 : 37 = 6·2 : 7·4.

From the above discussion we can say, if the distances of two other places from one place are shown by a cm and b cm in the wall map and by c cm and d cm respectively in the outline of the map, we get $a : b :: c : d$.

This shows that we can use the ideas of 'Ratio' and 'Proportion' in geometry also.

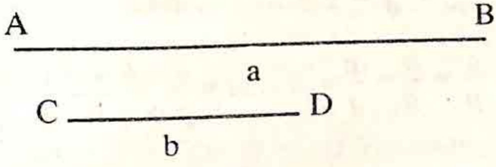


Fig. 166

If 'a' and 'b' be the lengths of two line segments AB and CD respectively, then

$$\frac{\text{Length of the line segment } AB}{\text{Length of the line segment } CD} = \frac{a}{b}.$$

We write the ratio of these two line segments as $\frac{AB}{CD}$.

We have obtained some properties of Proportion in arithmetic and algebra. These are also applicable to geometry. The properties are

1. If a, b, c, d be the lengths of four line segments and if they are proportional, that is $\frac{a}{b} = \frac{c}{d}$, then

$$\frac{b}{a} = \frac{d}{c} \quad .. \quad (i)$$

$$\frac{a}{c} = \frac{b}{d} \quad .. \quad (ii)$$

$$a.d = b.c \quad .. \quad (iii)$$

$$\frac{a+b}{b} = \frac{c+d}{d} \quad .. \quad (iv)$$

$$\frac{a-b}{b} = \frac{c-d}{d} \quad .. \quad (v)$$

$$\frac{a+b}{a-b} = \frac{c+d}{c-d} \quad .. \quad (vi)$$

2. If $\frac{a}{p} = \frac{b}{q} = \frac{c}{r} = \ldots\ldots$, then

$$\frac{a}{p} = \frac{b}{q} = \frac{c}{r} = \ldots\ldots = \frac{a+b+c+\ldots\ldots}{p+q+r+\ldots\ldots} \quad (i)$$

$$= \frac{la+mq+nc+\ldots\ldots}{lp+mq+nr+\ldots\ldots} \quad (ii)$$

3. If a, b, c be in continued proportion,

$$\frac{a}{b} = \frac{b}{c} \quad .. \quad (i)$$

$$or,\ ac = b^2 \quad .. \quad (ii)$$

[*Remarks:* (1) The geometrical significance of 1 *(iii)* is the rectangles contained by the line segments a, d and by the line segments b, c are equal in area.

(2) The geometrical significance of 3 *(ii)* is the rectangle with sides a, c, and the square with side b are equal in area. We can also say that the line segment **b** is mean proportional between the line segments a and c.

(3) The properties of ratio and proportion are applicable to the areas of geometrical figures just as they are applicable to the lengths of line segments.]

We have seen in arithmetic that if a quantity be divided into two parts in a definite ratio, we can find those two parts.

For example, 'if Rs. 50 be divided in the ratio 3 : 7, what amount will each get? The answer is, one will get Rs. 15 and the other Rs. 35.

You know, we can denote numbers by lengths of line segments. If we represent Rs. 50 by a line segment AB of length 10 cm and if we take a point C on AB such that

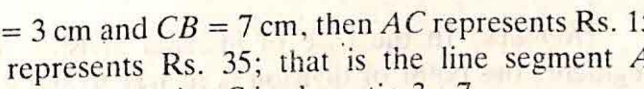


Fig. 167

$AC = 3$ cm and $CB = 7$ cm, then AC represents Rs. 15 and CB represents Rs. 35; that is the line segment AB is divided at the point C in the ratio 3 : 7.

But we are to think in a different way for division of a line segment. We know that a line segment can be produced. As a result, the point of division may be within the line segment as in the above case; the point of division may also be outside the line segment.

Let AB be a line segment of length 6 cm. It is to be divided in the ratio 3 : 1. If we take a point C within the line segment such that $AC = 4{\cdot}5$ cm and $CB = 1{\cdot}5$ cm, the line segment AB is divided at the point C in the ratio 3 : 1.

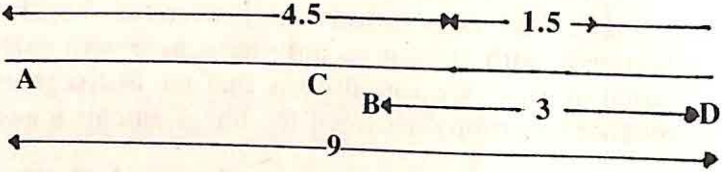


Fig. 168

Again had we produced AB to D such that $BD = 3$ cm, then AD will be 9 cm, that is $\frac{AD}{BD} = \frac{3}{1}$ In this case also, AB is divided at the point D in the ratio 3 : 1. In the first case, the sum of the two parts and in the second, their difference is the whole line segment. Thus we see if a line segment be divided in a definite ratio, the point of division may be within or outside the line segment. In the first case, that is, when the point of division is within the line segment, we say that the line segment is divided *internally* in that ratito. In the second case, that is, when the point of division is outside the line segment (but on that line). then we say that the line segment is divided *externally* in that ratio.

But there is a definite ratio in which a line segment can only be divided internally, but not externally. Say, what this ratio is.

[*Remark:* In the case of internal division of a line segment, the point of division is nearer to the extremity corresponding the lesser term of the ratio and for external division the line segment is to be produced to that extremity. In the above example, for internal division the point C is nearer to the point B and for external division, the point D lies on AB produced. Had the line segment been divided in the ratio 1 : 3, the point C would have been nearer to the end point A and D, on BA produced.]

5.2 A useful geometrical property :

If the bases of two triangles lie on the same straight line and the corresponding two vertices of the triangles be the same point, then the ratio of the areas of two triangles is equal to the ratio of their bases.

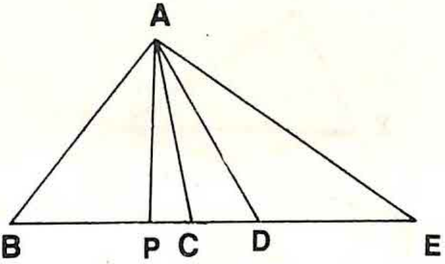


Fig. 169

Let the bases BC and DE of $\triangle^S$ ABC and ADE lie on the same straight line BE and A is the vertex of both the triangles.

To prove that $\dfrac{\triangle ABC}{\triangle ADE} = \dfrac{BC}{DE}$.

Construction: The perpendicular AP is drawn to BE from the point A.

Proof: $\because$ $AP \perp BE$ and only one perpendicular can be drawn to a straight line from an exterior point.

$\therefore$ $AP \perp$ BC and $AP \perp DE$.

$\therefore$ AP is the altitude of both the triangles.

$$\therefore \triangle ABC = \tfrac{1}{2}\, BC.\ AP$$

and $\triangle ADE = \tfrac{1}{2}\, DE.\ AP$

$$\therefore \frac{\triangle ABC}{\triangle ADE} = \frac{\frac{1}{2}\, BC.\ AP}{\frac{1}{2}\, DE.\ AP}$$

$$= \frac{BC}{DE}$$

5.3 We will use the above property to prove the next two theorems.

Theorem 46

If a straight line be drawn parallel to one side of a triangle, the other two sides are divided proportionally by that straight line.

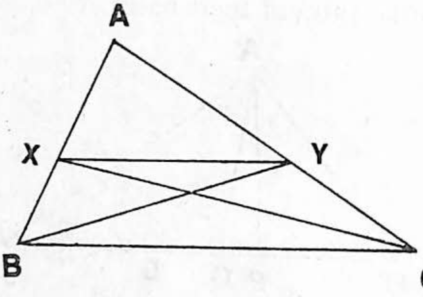


Fig. 170

Let ABC be a triangle. A straight line parallel to the side BC meets AB and AC at the points X and Y respectively.

To prove that $\frac{AX}{XB} = \frac{AY}{YC}$.

Construction: B, Y and C, X are joined.

Proof: The bases AX, XB of Δ^s AYX and BYX lie on the same straight line and they have the same vertex Y,

$$\therefore \quad \frac{\Delta AYX}{\Delta BYX} = \frac{AX}{XB}.$$

Again, the bases AY, YC of Δ^s AXY and CXY lie on the same straight line and they have the same vertex X,

$$\therefore \quad \frac{\Delta AXY}{\Delta CXY} = \frac{AY}{YC}.$$

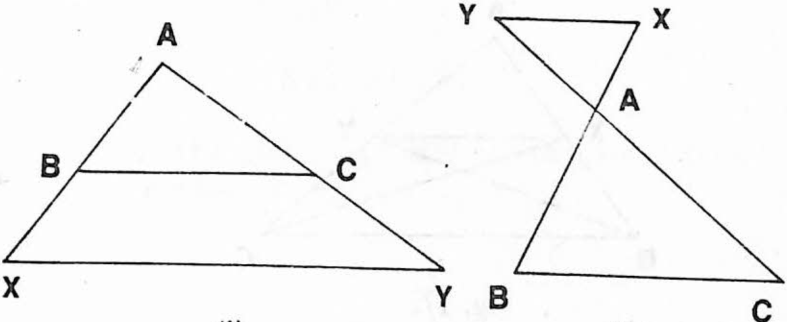
ΔBYX and ΔCXY stand on the same base XY and between the same parallels XY and BC,

$$\Delta BYX = \Delta CXY.$$

$$\therefore \quad \frac{\Delta AYX}{\Delta BXY} = \frac{\Delta AXY}{\Delta CXY}$$

$$\therefore \quad \frac{AX}{XB} = \frac{AY}{YC}$$

(*Remarks:* (1) In Fig. 170 the straight line XY divides AB and AC internally; XY could have divided AB and AC externally [Fig. 171(i)] or could have also divided BA and CA externally [Fig. 171(ii)].



(i) (ii)

Fig. 171

In both these cases, the theorem could have been proved in the same way. Try for yourself.

(2) We have proved, in Fig. No. 170, that

$$\therefore \frac{AX}{XB}=\frac{AY}{YC}$$

Hence, we also get *(i)* $\frac{AB}{XB}=\frac{AC}{YC}$ and *(ii)* $\frac{AB}{AX}=\frac{AC}{AY}$

(i) $\because \frac{AX}{XB}=\frac{AY}{YC}$

$\therefore \frac{AX}{XB}+1=\frac{AY}{YC}+1$

or, $\frac{AX+XB}{XB}=\frac{AY+YC}{YC}$

or, $\frac{AB}{XB}=\frac{AC}{YC}$

(ii) $\because \frac{AX}{XB}=\frac{AY}{YC}$

$\therefore \frac{XB}{AX}=\frac{YC}{AY}$

or, $1+\frac{XB}{AX}=1+\frac{YC}{AY}$

or, $\frac{AX+XB}{AX}=\frac{AY+YC}{AY}$

or, $\frac{AB}{AX}=\frac{AC}{AY}$

(3) Here $\triangle AYX$ denotes the area of the triangle AYX.]

Theorem 47

The straight line that divides the two sides of a triangle proportionally, is parallel to the third side.

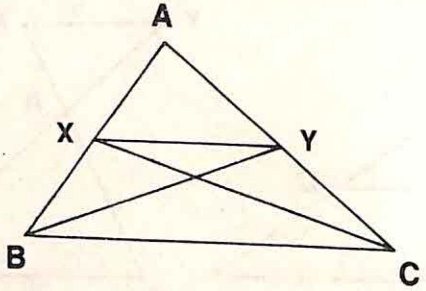


Fig. 172

Let ABC be a triangle. Let the straight line XY divide the sides AB and AC proportionally at the points and Y respectively; that is

$$\frac{AX}{XB}=\frac{AY}{YC}$$

To prove that $XY \parallel BC$.
Construction: B, Y and C, X are joined.

Proof: The bases AX and XB of $\triangle AYX$ and $\triangle BYX$ lie on the same straight line and they have the same vertex Y.

$$\therefore \frac{\triangle AYX}{\triangle BYX}=\frac{AX}{XB}$$

Again, the bases AY and YC of $\triangle AXY$ and $\triangle CXY$ lie on the same straight line and they have the same vertex X,

$$\therefore \frac{\triangle AXY}{\triangle CXY}=\frac{AY}{YC}$$

But, $\dfrac{AX}{XB}=\dfrac{AY}{YC}$ (by hypothesis)

$$\therefore \frac{\triangle AYX}{\triangle BYX}=\frac{\triangle AXY}{\triangle CXY}$$

$$\therefore \triangle BYX=\triangle CXY$$

But $\triangle BYX$ and $\triangle CXY$ stand on the same base XY and on the same side of it.

$$\therefore\ XY \parallel BC.$$

Alternative Proof:

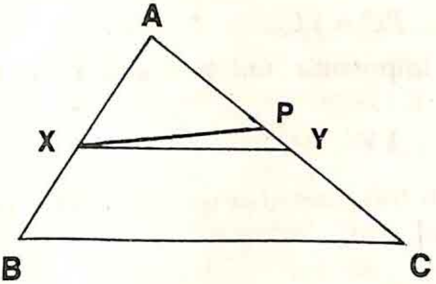


Fig. 173

Let ABC be a triangle. Let the straight line XY divide the sides AB and AC proportionally at the points X and Y respectively; that is,

$$\frac{AX}{XB}=\frac{AY}{YC}$$

To prove that $XY \parallel BC$.

Proof: If XY and BC be not parallel, then let us suppose that the straight line drawn parallel to BC through the point X meets AC at the point P.

$$\therefore \frac{AX}{XB}=\frac{AP}{PC}$$

But, $\dfrac{AX}{XB} = \dfrac{AY}{YC}$ (By hypothesis),

$$\therefore \frac{AP}{PC} = \frac{AY}{YC}$$

$$\therefore \frac{AP}{PC} + 1 = \frac{AY}{YC} + 1$$

$$\text{or, } \frac{AP + PC}{PC} = \frac{AY + YC}{YC}$$

$$\text{or, } \frac{AC}{PC} = \frac{AC}{YC}$$

$$PC = YC.$$

But this is impossible unless P and Y are the same point.

$$\therefore XY \parallel BC.$$

[Remark: In this case also we could have two other different figures.]

Some Applications:

1. In $\triangle ABC$, the straight line parallel to the side BC meets AB and AC at the points D and E respectively. If $AE = 2AD$, find $DB : EC$.

Answer: $\because DE \parallel BC$

$$\therefore \frac{AD}{DB} = \frac{AE}{EC}$$

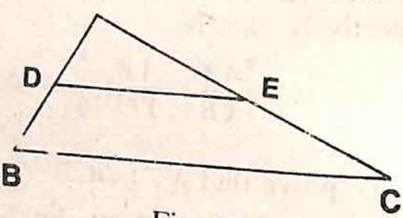


Fig. 174

that is $\dfrac{AD}{AE} = \dfrac{DB}{EC}$

But $\dfrac{AD}{AE} = \frac{1}{2}$ ($\because$ $AE = 2AD$),

$$\therefore DB : EC = 1 : 2.$$

2. In trapezium $ABCD$, $AB \parallel DC$. A straight line parallel to AB meets AD and BC at the points E and F respectively. Prove that $AE : ED = BF : FC$.

Construction: A, C are joined. AC intersects EF at the point G.

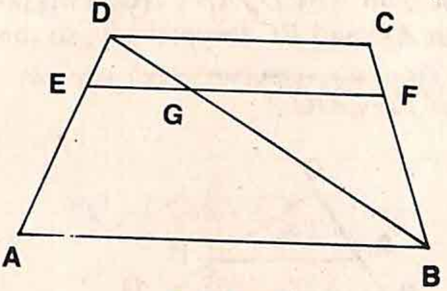


Fig. 175

Proof: In $\triangle ADC$, EG is parallel to the side DC,

$\therefore AE : ED = AG . GC$.

Again, in $\triangle ACB$, GF is parallel to the side AB,

$\therefore AG : GC = BF ; FC$.

$\therefore AE : ED = BF : FC$.

3. $ABCD$ is a quadrilateral. Three points P, Q, R are taken on two sides AB, AD and the diagonal AC respectively so that $PR \parallel BC$ and $RQ \parallel CD$. Prove that $PQ \parallel BD$.

Proof:

$\because PR \parallel BC$

$\therefore$ we get from $\triangle ABC$,

$$\frac{AP}{PB} = \frac{AR}{RC}.$$

Again, $\because RQ \parallel CD$

$\therefore$ we get from $\triangle ACD$,

$$\frac{AR}{RC} = \frac{AQ}{QD}$$

$$\therefore \frac{AP}{PB} = \frac{AQ}{QD}$$

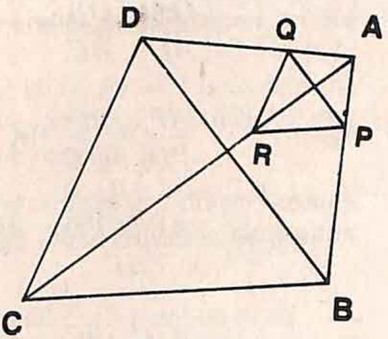


Fig. 176

$\therefore$ we get from $\triangle ABD$,

$$PQ \parallel BD.$$

4. In trapezium $ABCD$, $AB \parallel DC$. Two points P and Q are taken on AD and BC respectively, so that $AP : PD = BQ : QC$.

Prove that $PQ \parallel DC$.

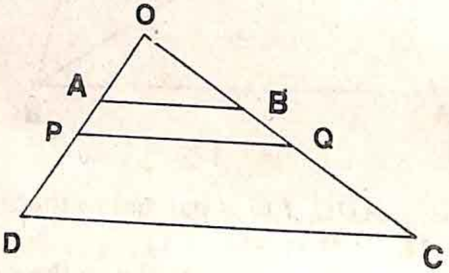


Fig. 177

Construction: Let $AB<DC$. Both DA and CB, when produced, meet at the point O. P, Q are joined.

Proof: $\because AB \parallel DC$,

$\therefore$ We get from $\triangle ODC$,

$$\frac{OA}{AD} = \frac{OB}{BC}$$

Again, $\frac{AP}{PD} = \frac{BQ}{QC}$ (by hypothesis)

that is, $\frac{AP+PD}{AP} = \frac{BQ+QC}{BQ}$

or, $\frac{AD}{AP} = \frac{BC}{BQ}$

$$\therefore \frac{OA}{AD} \times \frac{AD}{AP} = \frac{OB}{BC} \times \frac{BC}{BQ}$$

or, $$\frac{OA}{AP} = \frac{OB}{BQ}$$

We get from $\triangle OPQ$, $AB \parallel PQ$.

But $AB \parallel DC$,

$\therefore PQ \parallel DC$.

Exercise 24

1. In $\triangle ABC$, the straight line parallel to the side BC meets AB and AC at the points P and Q respectively. *(i)* If $PB = AQ$, $AP = 9$ units, $QC = 4$ units. How many units is the length of PB? *(ii)*The length of PB is twice the length of AP, the length of QC exceeds the length of AQ by 3 units. How many units is the length of AC? *(iii)* $AP = QC$; the length of AB is 12 units and the length of AQ is 2 units. How many units is the length of CQ?

2. In $\triangle PQR$, X, Y are two points on the sides PQ and PR respectively.

(i) If $PX = 2$ units, $XQ = 3{\cdot}5$ units, $YR = 7$ units and $PY = 4{\cdot}25$ units, will XY and QR be parallel to each other.? Give reasons.

(ii) If $PQ = 8$ units, $YR = 12$ units, $PY = 4$ units and the length of PY is 2 units less than the length of XQ, will XY and QR be parallel to each other? Give reasons.

3. Prove that, in a triangle the straight line drawn through the mid-point of one side parallel to the second side bisects the third side.

4. In $\triangle ABC$, AB is a median and P is a point on it. BP and CP, when produced, meet AC and AB at the points Q and R respectively. Prove that $RQ \parallel BC$.

5. Prove that three parallel lines divide two transversals proportionally.

6. In $\triangle ABC$ the medians BE and CF intersect each other at the point G and the line segment FE intersects the line segment AG at the point O. Prove that $AO = 3OG$.

7. Prove that the line segment joining the two mid-points of oblique sides of a trapezium is parallel to the parallel sides.

8. Prove that the line segment joining the mid-points of two sides of a triangle is parallel to the third side.

9. $\triangle ABC$ and $\triangle DBC$ stand on the same base BC and on the same side of it. E is a point on BC. Through the point E parallels to BA and BD are drawn to meet AC and DC at the points F and G respectively. Prove that $AD \parallel FG$.

10. D is a point on the side BC of $\triangle ABC$. P, Q are centroids of $\triangle ABD$ and $\triangle ADC$ respectively. Prove that $PQ \parallel BC$.

11. OAP, OBQ and OCR are three line segments. The points A, P; B, Q and C, R are such that $AB \parallel PQ$ and $BC \parallel QR$. Prove that $AC \parallel PR$.

12. $\triangle PQR$ and $\triangle SQR$ stand on the same base QR and on the same side of it. Two triangles are equal in area. F and G are centroids of two triangles respectively. Prove that $FG \parallel QR$.

13. Prove that the angles adjacent to any one of the parallel sides of an isosceles trapezium are equal.

14. If the angles adjacent to any one of the parallel sides of a trapezium be equal, the trapezium is isosceles.

5.4 You know that two triangles, with their three angles equal, are equiangular.

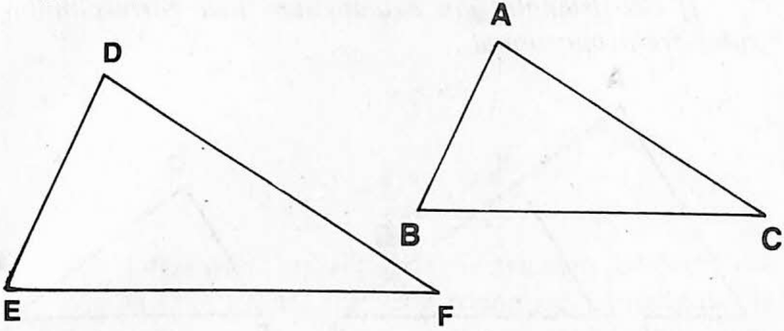


Fig. 178

In Fig. 178, in $\triangle ABC$ and $\triangle DEF$, $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$. Therefore, $\triangle ABC$ and $\triangle DEF$ are equiangular.

[*Remarks:* (1) As the sum of the three angles of a triangle is equal to two right angles, then two angles of a triangle being equal to two angles of another, the third angles must be equal; that is, two triangles may be said to be equiangular if two angles of one are equal to two angles of the other.

(2) Two equilateral triangles must be equiangular.]

Theorem 48

If two triangles are equiangular their corresponding sides are proportional.

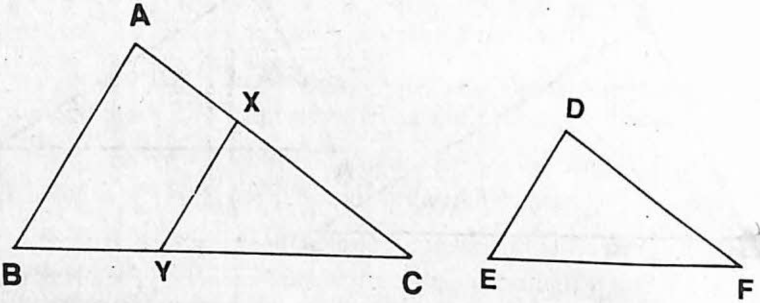


Fig. 179

Let $\triangle ABC$ and $\triangle DEF$ be two equiangular triangles, so that $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$.

To prove that $\dfrac{BC}{EF} = \dfrac{CA}{FD} = \dfrac{AB}{DE}$

Construction: From CA (or CA produced) CX is cut off making equal to FD and from CB (or CB produced) CY is cut off making equal to FE, X, Y are joined.

Proof: In $\triangle CXY$ and $\triangle FDE$

$CX = FD$ (by construction),

$CY = FE$ (by construction),

included $\angle XCY$ = included $\angle DFE$ (by hypothesis).

$\therefore \triangle CXY \cong \triangle FDE$.

$\therefore \angle CXY = \angle FDE$ (corresponding angles)

$= \angle CAB$ (by hypothesis).

$\therefore XY \parallel AB$ ($\because \angle CXY$ and $\angle CAB$ are corresponding angles).

$$\therefore \frac{CB}{CY} = \frac{CA}{CX}$$

that is, $\dfrac{BC}{EF} = \dfrac{CA}{FD}$ ($\because CY = FE$, $CX = FD$).

Similarly, by cutting off a part equal to *DE* from *AB* (or *AB* produced) and a part equal to *DF* from *AC* (or *AC* produced) and joining the points of division, it can be proved that

$$\frac{CA}{FD} = \frac{AB}{DE}$$

$$\therefore \frac{BC}{EF} = \frac{CA}{FD} = \frac{AB}{DE}$$

[*Remark:* Observe that the triangle formed by drawing parallel to one side of a triangle is equiangular to the original triangle. Hence, their corresponding sides are proportional.

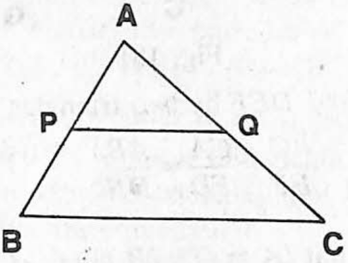


Fig. 180

In Fig. 180, $PQ \parallel BC$,

$\therefore$ in $\triangle APQ$ and $\triangle ABC$,

$$\angle APQ = \text{corresponding } \angle ABC,$$

$$\angle AQP = \text{corresponding } \angle ACB,$$

$\therefore \triangle APQ$ and $\triangle ABC$ are equiangular.

$$\therefore \frac{AQ}{AC} = \frac{AP}{AB} = \frac{PQ}{BC} \;]$$

Theorem 49

It the sides of two triangles are proportional, the corresponding angles of the two triangles are equal, that is the two triangles are equiangular.

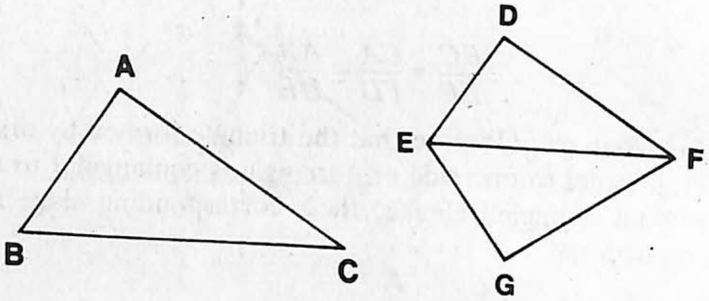


Fig. 181

Let ABC and DEF be two triangles, so that

$$\frac{BC}{EF}=\frac{CA}{FD}=\frac{AB}{DE}$$

To prove that $\angle A=\angle D, \angle B=\angle E, \angle C=\angle F$, that is, $\triangle ABC$ and $\triangle DEF$ are equiangular.

Construction: A triangle EFG is constructed with EF as one of its sides such that the point G is on the opposite side of the point D and $\angle GEF=\angle ABC, \angle GFE=\angle ACB$.

Proof: In $\triangle ABC$ and $\triangle GEF$,

$\angle ABC=\angle GEF$ (by construction),

$\angle ACB=\angle GFE$ (by construction),

$\therefore$ the remaining $\angle BAC=$ the remaining $\angle EGF$.

$\therefore$ $\triangle ABC$ and $\triangle GEF$ are equiangular.

$$\therefore \frac{CA}{FG}=\frac{AB}{GE}=\frac{BC}{EF}$$

$$\therefore \frac{CA}{FG}=\frac{CA}{FD}\left(\because \text{ each of them is equal to } \frac{BC}{EF}\right)$$

$$\therefore FG=FD$$

Similarly, $GE = DE$.

Again, in $\triangle DEF$ and $\triangle GEF$,

$DE = GE$ (proved),

$DF = GF$ (proved),

and EF is common.

$\therefore\ \triangle DEF \cong \triangle GEF$.

$\therefore\ \angle EDF = \angle EGF = \angle BAC$,

$\angle DEF = \angle GEF = \angle ABC$,

and $\angle DFE = \angle GFE = \angle ACB$,

that is, $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$;

$\therefore\ \triangle ABC$ and $\triangle DEF$ are equiangular.

[*Remarks:* (1) As in the proof of Theorem 48, theorem 49 also could have been proved by cutting off from two sides of $\triangle ABC$ two parts equal to the corresponding sides of $\triangle DEF$. Try for yourself.

(2) If two sides of a triangle be divided proportionally, the triangle formed by joining the points of division is equiangular to the original triangle; that is, in Fig. 182 $\triangle APQ$ and $\triangle ABC$ are equiangular.

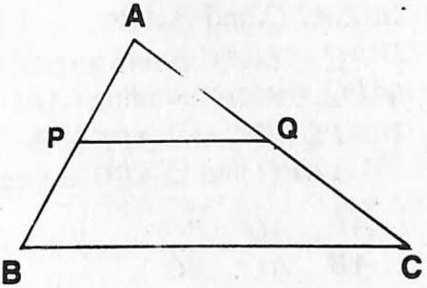


Fig. 182

$$\because \frac{AP}{PB} = \frac{AQ}{QC}$$

$$\therefore PQ \parallel BC$$

$\therefore \triangle APQ$ and $\triangle ABC$ are equiangular.

Also observe that $\dfrac{AP}{AB} = \dfrac{AQ}{AC} = \dfrac{PQ}{BC}$]

Some Applications:

1. Prove that, the straight line drawn through the mid-point of one side of a triangle parallel to another side bisects the third side and the intercept of the parallel straight line is half the second side.

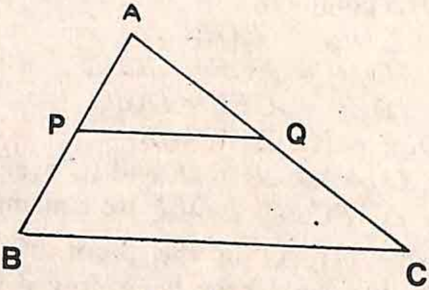


Fig. 183

Let ABC be a triangle and P be the mid-point of the side AB. The straight line drawn through the point P parallel to BC meets AC at the point Q.

To prove that *(i)* Q is the mid-point of AC,

(ii) $PQ = \frac{1}{2} BC$.

Proof: In $\triangle APQ$ and $\triangle ABC$,

$\angle PAQ = \angle BAC$ (same angle),

$\angle APQ =$ corresponding $\angle ABC$

($\because PQ \parallel BC$ and APB is the transversal)

$\therefore \triangle APQ$ and $\triangle ABC$ are equiangular.

$$\therefore \frac{AP}{AB} = \frac{AQ}{AC} = \frac{PQ}{BC}$$

But $\frac{AP}{AB} = \frac{1}{2}$ ($\because$ P is the mid-point of AB).

$$\therefore \frac{AQ}{AC} = \frac{1}{2};$$

that is, $AQ = \frac{1}{2} AC$

$\therefore$ Q is the mid-point *of* AC

and $\frac{PQ}{BC} = \frac{1}{2}$ ... *(i)*

$\therefore\ PQ = \frac{1}{2} BC.$ ... *(ii)*

[Observe that Theorem 23 is proved here in a different way.]

2. *AD* is a median of $\triangle ABC$. A straight line parallel to *BC* meets *AB* and *AC* at the points *P* and *Q* respectively. Prove that the line segment *PQ* is bisected by *AD*.

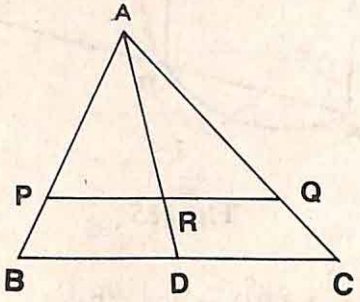


Fig -184

Let *PQ* intersect *AD* at the point *R*.
To prove that $PR = RQ$

Proof: In $\triangle APR$ and $\triangle ABD$,
$\angle PAR = \angle BAD$ (same angle),
$\angle APR =$ corresponding $\angle ABD$
($\because\ PR \parallel BD$ and *APB* is the transversal).
$\therefore\ \triangle APR$ and $\triangle ABD$ are equiangular.

$$\therefore \frac{PR}{BD} = \frac{AR}{AD}$$

Similarly, it can be proved that $\frac{RQ}{DC} = \frac{AR}{AD}$

$$\therefore \frac{PR}{BD} = \frac{RQ}{DC}$$

But $BD = DC$ ($\because$ *AD* is the median),
$\therefore\ PR = RQ$

3. *ABCD* is a cyclic quadrilateral. *AB* and *DC*, when produced, meet each other at the point *P*. Prove that, $PA.PB = PC.PD$.

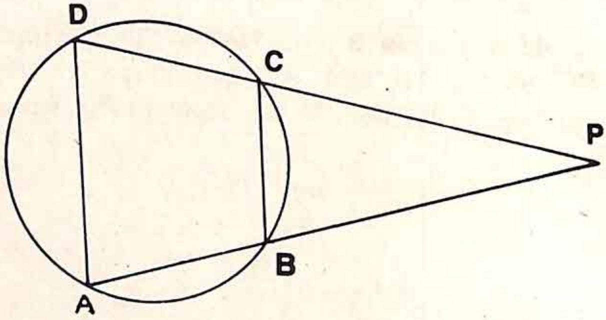


Fig. 185

Proof: In $\triangle APD$ and $\triangle CPB$,

$\angle APD = \angle CPB$ (same angle),

$\angle PAD = \angle BCP$

($\because$ *ABCD* is a cyclic quadrilateral)

$\therefore$ $\triangle APD$ and $\triangle CPB$ are equiangular.

$$\therefore \frac{PA}{PC} = \frac{PD}{PB}$$

$$\therefore PA.PB = PC.PD.$$

(*Remark:* If we write the relation $PA.PB = PC.PD$ in the form of ratios, we can write $\frac{PA}{PC} = \frac{PD}{PB}$. From this it is clear that we can prove the property, if we can show $\triangle APD$ and $\triangle CPB$ equiangular.

Again, since the side *PC* corresponds to the side *PA* and the side *PB* corresponds to the side *PD*, we will realise which angles are to be shown equal.]

4. *ABCD* is a cyclic quadrilateral. Prove that $AC.BD = AB.DC + AD.BC$.

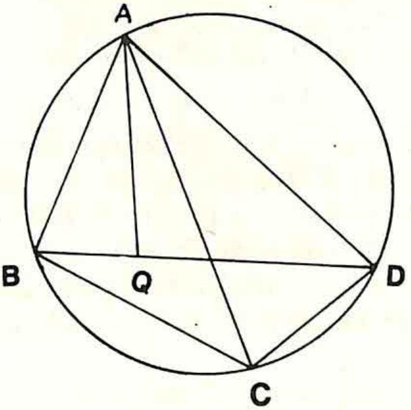


Fig. 186

Construction: A point Q is taken on BD, so that $\angle DAQ = \angle CAB$.

Proof: In $\triangle DAQ$ and $\triangle BAC$

$\angle DAQ = \angle BAC$ (by construction),

$\angle ADQ = \angle ACB$ (angles in the same segment),

$\therefore$ $\triangle DAQ$ and $\triangle BAC$ are equiangular.

$$\therefore \frac{AD}{AC} = \frac{DQ}{BC}$$

$\therefore AC.DQ = AD.BC.$

$\because \angle BAC = \angle DAQ$ (by construction).

$\therefore \angle BAC - \angle CAQ = \angle DAQ - \angle CAQ.$

$\therefore \angle BAQ = \angle DAC.$

Now, in $\triangle BAQ$ and $\triangle CAD$,

$\angle BAQ = \angle DAC$

$\angle ABQ = \angle ACD$ (angles in the same segment);

$\therefore \triangle BAQ$ and $\triangle CAD$ are equiangular.

$$\therefore \frac{AB}{AC} = \frac{BQ}{DC}$$

$$\therefore AC.BQ = AB.DC.$$
$$\therefore AC.BQ + AC.DQ = AB.DC + AD.BC.$$
$$\therefore AC.(BQ + QD) = AB.DC + AD.BC.$$
$$\therefore AC.BD = AB.DC + AD.BC.$$

[*Remark:* Since it is to be shown that $AC.BD = AB.DC + AD.BC$, BD is divided in two parts taking a point Q on BD, so that we can get $AC.BD = AC.BQ + AC.DQ$. Now, one of them is shown to be equal to $AB.DC$; that is, we are to get two equiangular triangles. Keeping this in mind, we are to fix the position of the point Q.

In the same way, it could have been proved by taking a point on AC and not the point Q on BD. Try for yourself.

This is known as Ptolemy's Theorem.]

Exercise 25

1. Prove that the line segment joining mid-points of two sides of a triangle is parallel to and half the thrid side.

2. Two parallel straight lines intersect three concurrent lines at the points A, B, C and X, Y, Z respectively. Prove that $AB:BC = XY:YZ$.

3. In a trapezium $PQRS$, $PQ \parallel SR$. Two diagonals PR and QS intersect each other at the point O. Prove that $OP:OR = OQ:OS$.

If $SR = 2PQ$, prove that O is one of the points of trisection of each of the diagonals.

4. $PQRS$ is a parallelogram. A straight line through the point S intersects PQ and RQ produced at the points X and Y respectively. Prove that $PS:PX = QY:QX = RY:RS$.

5. ABC is an equilateral triangle, the length of each side being a units. BC is produced to the point P and CB to the point Q, so that $BQ = CP = a$. Join A, P and A, Q; and prove that *(i)* $PQ:PA = PA:PC$, *(ii)* $PA^2 = 3a^2$.

6. Two acute-angled triangles $\triangle ABC$ and $\triangle PQR$ are equiangular. Their circum-centres are X and Y respectively. If BC and QR be two corresponding sides, prove that $BX:QY = BC:QR$.

7. Two chords PQ and RS of a circle intersect each other internally at the point X. Join P, S and R, Q; and prove that $\triangle PXS$ and $\triangle RXQ$ are equiangular. Hence, prove that $PX.XQ = RX.XS$

5.5. Idea of similar triangles and some other similar plane figures :

In the study on Ratio and Proportion, you must have seen that a wall-map and an outline of a map are of same shape but look small or large in size. The distances of two corresponding points on those two maps always bear the same ratio.

Let us take another example. Suppose, the floor of your class-room is rectangular and its length is 6·8 metres and breadth 5·1 metres. If you want to draw a sketch of that floor in your exercise book, you will have to draw a rectangle. Its size will surely be smaller than that of the floor of the room. But can you take the length and breadth of any measurement? Surely not.

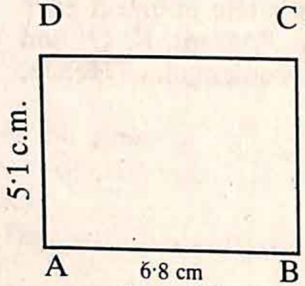


Fig. 187

If the length of the rectangle, you draw, be 6·8 cms, you are to take breadth as 5·1 cms (Fig. 187).

If one draws the sketch on a paper larger in size than the paper on which you have drawn, he can draw the rectangle of much bigger size. If he takes 13·6 cms as length, he will have to take 10·2 cms as breadth (Fig. 188).

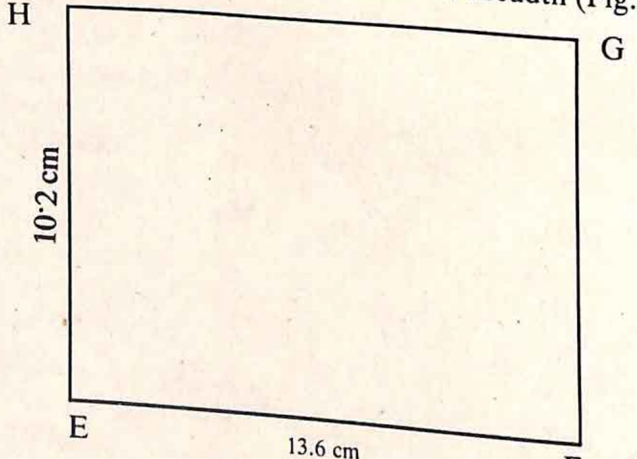


Fig. 188

Again, if one wants to draw a smaller figure, the length and breadth of the rectangle drawn by him will be of smaller measurements, but the relatiton between the length and breadth will be the same as before; that is, if he takes the length as 4 cms, he will have to take breadth as 3 cms, (Fig. 189).

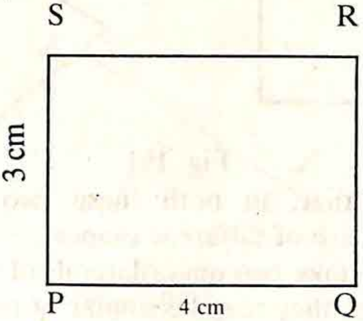


Fig. 189

So observe, these three sketches are of the same shape, that is, all these three are rectangles.

Angles of these three figures are equal (each of them is a right angle) and also the ratios of the corresponding sides are equal, which means the corresponding sides are proportional. Such figures are called *similar figures*. So, similar polygons may be defined as follows:

Definition: Those polygons, which are equiangular (that is, their angles are equal) and whose corresponding sides are proportional, are similar.

[*Remarks:* (1) To investigate the similarity of two polygons, we are to see whether they satisfy these two conditions. For example, *(i)* a square and a rectangle are equiangular, but their corresponding sides are never proportional. Hence they are not similar.

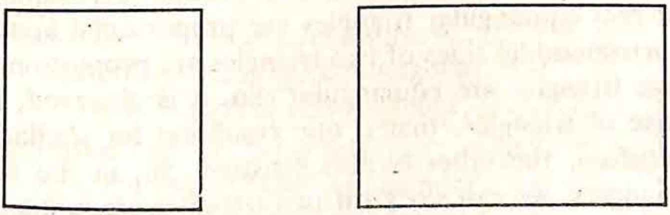


Fig. 190

(ii) the corresponding sides of a square and a rhombus are always proportional but they are never equiangular (for, each angle of a square is a right angle but none of a rhombus is a right angle), hence they are not similar.

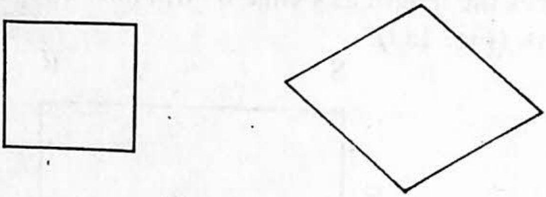


Fig. 191

Observe that, in both these two cases the two quadrilaterals are of different shapes.

(iii) If we take two quadrilaterals of same shape, say, two rectangles, they may be similar or not.

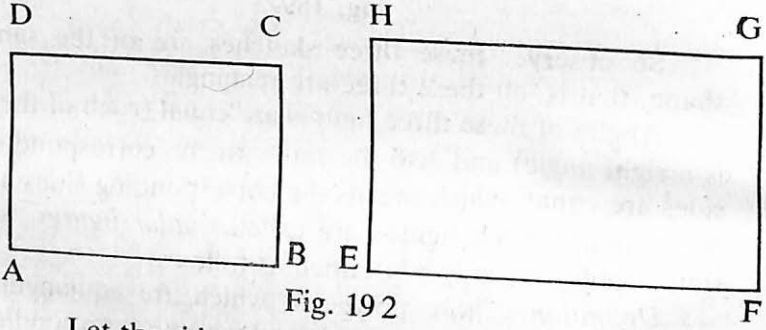


Fig. 192

Let the sides AB and AD of the rectangle $ABCD$ be of lengths 5 cms and 3 cms. Again, let the sides EF and EH of the rectangle $EFGH$ be of lengths 7 cms and 4 cms. It is clear that they are equiangular but their corresponding sides are not proportional. Hence they are not similar.

(2) But in the case of triangles, we have proved in Theorem 48 and Theorem 49, that the corresponding sides of two equiangular triangles are proportional and if the corresponding sides of two triangles are proportional, the two triangles are equiangular. So, it is observed, in the case of triangles, that if one condition for similarity be satisfied, the other is also satisfied. So, in the case of triangles, we can say *(i)* if two triangles are equiangular, they are similar, or, *(ii)* if sides of two triangles are proportional, the triangles are similar.]

Theorem 50

If a perpendicular is drawn from the vertex containing the right angle of a right-angled triangle on the hypotenuse, the triangles on each side of the perperndicular are similar to the whole triangle and to one another.

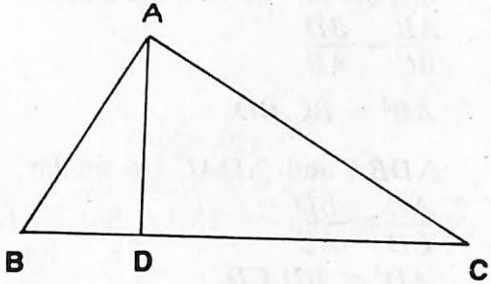


Fig. 193

Let ABC be a triangle right-angled at A and AD be perpendicular on the hypotenuse BC from A, the vertex containing right angle.

To prove that *(i)* $\triangle DBA$ and $\triangle ABC$ are similar.
(ii) $\triangle DAC$ and $\triangle ABC$ are similar.
(iii) $\triangle DBA$ and DAC are similar.

Proof : $\angle BDA = \angle BAC = \angle ADC$ (each being right angle),

$\therefore \angle BAD = \angle BCA = \angle ACD$ (complement of $\angle ABC$ or $\angle ABD$)

and $\angle ABD = \angle CBA = \angle CAD$ (complement of $\angle ACB$ or $\angle ACD$).

From first and second relations, we get,

$\triangle DBA$ and $\triangle ABC$ are similar.

$\therefore$ $\triangle DBA$ and $\triangle ABC$ are similar. .. *(i)*

From second and third relations, we get,

$\triangle ABC$ and $\triangle DAC$ equiangular,

$\therefore$ $\triangle ABC$ and $\triangle DAC$ are similar. .... *(ii)*

And from first and third relations, we get

$\triangle DBA$ and $\triangle DAC$ equiangular,

$\therefore$ $\triangle DBA$ and $\triangle DAC$ are similar. .... *(iii)*

Corollary 1 From Fig. 193, we get

(i) $AB^2 = BC.BD$.

(ii) $AD^2 = BD.CD$.

(iii) $AC^2 = BC.CD$.

(i) $\because \triangle DBA$ and $\triangle ABC$ are similar,

$$\therefore \frac{AB}{BC} = \frac{BD}{AB}$$

$$\therefore AB^2 = BC.BD.$$

(ii) $\because \triangle DBA$ and $\triangle DAC$ are similar,

$$\therefore \frac{AD}{CD} = \frac{BD}{AD}.$$

$$\therefore AD^2 = BD.CD.$$

(iii) $\because \triangle DAC$ and $\triangle ABC$ are similar,

$$\therefore \frac{AC}{BC} = \frac{DC}{AC}$$

$$\therefore AC^2 = BC.DC.$$

[*Remarks:* (1) Observe that from the vertex containing the right angle three line segments are drawn to meet the hypotenuse at three points respectively. As a result three line segments are obtained on the hypotenuse. Any of the three line segments is mean proportional between two line segments of the hypotenuse so that one end point of these two line segments is the point of intersection of the first line segment with the hypotenuse.

(2) From the three corollaries we get AB, AD and AC are mean proportionals between BC, BD; BD, CD and BC, CD respectively; that is, the squares on sides AB, AD and AC are equal in area to the rectangles containing the sides BC and BD, BD and CD, and BC and CD respectively.]

Corollary 2. If one pair of acute angles of two right-angled triangles be equal, the other pair will also be equal, that is the two triangles are equiangular. Hence the corresponding sides of these two triangles are proportional.

In Fig. 194, let $\triangle ABC$ and $\triangle DEF$ are right-angled triangles. Let $\angle C$ and $\angle F$ be right angles and let $\angle B$ and $\angle E$

Fig. 194

be also equal. Therefore, $\angle A = \angle D$.

$\triangle ABC$ and $\triangle DEF$ are equiangular.

$$\therefore \frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$

Some Applications:

1. AB is a diameter of a circle. From any point P on the circle, the perpendicular drawn on AB meets AB at the point N. Prove that $PB^2 = AB.BN$.

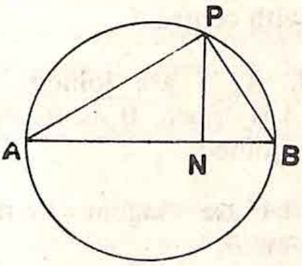


Fig. 195

Proof: $\because$ AB is a diameter,

$\therefore$ $\angle APB$ is an angle in the semi-circle.

$\therefore$ $\angle APB = 1$ right angle.

$\therefore$ PN is perpendicular on the hypotenuse AB from the vertex P containing right angle.

$\therefore$ $\triangle ABP$ and $\triangle PBN$ are similar.

$$\therefore \frac{PB}{NB} = \frac{AB}{PB}.$$

$$\therefore PB^2 = AB.BN.$$

2. Two circles touch each other externally at the point A. PQ is a direct common tangent of those two circles. If r and r' be the radii of those two circles respectively, prove that $PQ^2 = 4rr'$.

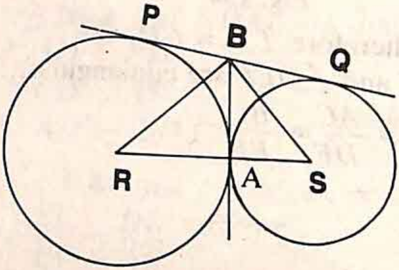


Fig. 196

Let the centre of one circle be R and that of the other be S. Let r be the radius of the circle with centre R and r' be that of the circle with centre S.

Construction: R, A, S are joined. The common tangent at the point A is drawn. It meets PQ at the point B. B, R and B, S are joined.

Proof: $\because$ BP and BA are tangents to the circle with centre R from the point B,

$\therefore$ $PB = AB$

and BR is the bisector of $\angle ABP$.

Similarly, $BQ = AB$

and BS is the bisector of $\angle ABQ$.

$\therefore$ $\angle RBS = 1$ right angle.

$\because$ R, S are the centres of two circles and A is the point of contact,

$\therefore$ R, A, S are collinear and $AB \perp RS$.

$\therefore$ in the right-angled triangle RBS, BA is the perpendicular on the hypotenuse RS from the vertex B containing the right angle.

$$\therefore \triangle ABR \text{ and } \triangle ASB \text{ are similar.}$$

$$\therefore \frac{AB}{AS} = \frac{AR}{AB}$$

$$\therefore AB^2 = AR.AS$$

$$= rr' \; (\therefore AR = r, AS = r').$$

$$\therefore 4\,AB^2 = 4\,rr'$$

$$\text{or } (2\,AB)^2 = 4rr'$$

$$\text{or } PQ^2 = 4\,rr' \; (\because PQ = PB + BQ = 2\,AB).$$

Exercise 26

1. AB is a diameter of a circle. A straight line through the point A meets the circle at the point C and the tangent, drawn at the point B, at the point D. Prove that *(i)* $BD^2 = AD.DC$, *(ii)* for any straight line, the rectangle contained by AC and AD will have the same area.

2. AB is a diameter of a circle with centre O. P is any point of the circle. The tangent drawn at the point P meets the tangents, drawn at the points A and B, at the points Q and R respectively. If r be the radius of the circle, prove that $PQ.PR = r^2$.

3. A semi-circle is drawn with AB as diameter. C is any point on AB. The perpendicular drawn on AB from the point C meets the semi-circle at the point D. Prove that CD is the mean proportional between AC and BC.

4. ABC is a triangle right-angled at A. AD is perpendicular on the hypotenuse BC. Prove that,

$$\frac{\triangle ABC}{\triangle ACD} = \frac{BC^2}{AC^2}$$

Pythagoras' Theorem and its converse :

Theorem 51

The area of the square on the hypotenuse of a right-angled triangle is equal to the sum of the areas of the squares on other two sides.

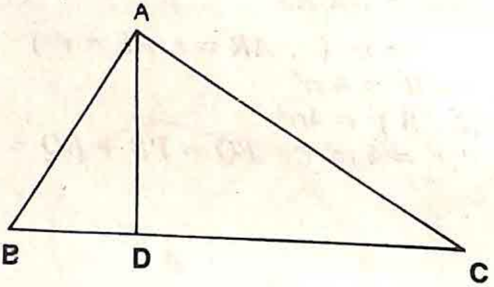


Fig. 197

Let ABC be a triangle right-angled at A.
To prove that, $BC^2 = AB^2 + AC^2$.

Construction : AD is drawn perpendicular on the hypotenuse BC from the vertex A containing the right angle.

Proof: $\because$ in right-angled triangle ABC, AD is perpendicular on the hypotenuse BC,

$$\therefore \triangle ABD \text{ and } \triangle CBA \text{ are similar.}$$

$$\therefore \frac{AB}{BC} = \frac{BD}{AB}.$$

$$\therefore AB^2 = BC.BD.$$

Again, $\triangle ACD$ and $\triangle CBA$ are also similar.

$$\therefore \frac{AC}{BC} = \frac{DC}{AC}.$$

$$\therefore AC^2 = BC.DC.$$

$$\begin{aligned} AB^2 + AC^2 &= BC.BD + BC.DC \\ &= BC(BD + DC) \\ &= BC^2. \end{aligned}$$

$$\therefore BC^2 = AB^2 + AC^2.$$

Alternative proof :

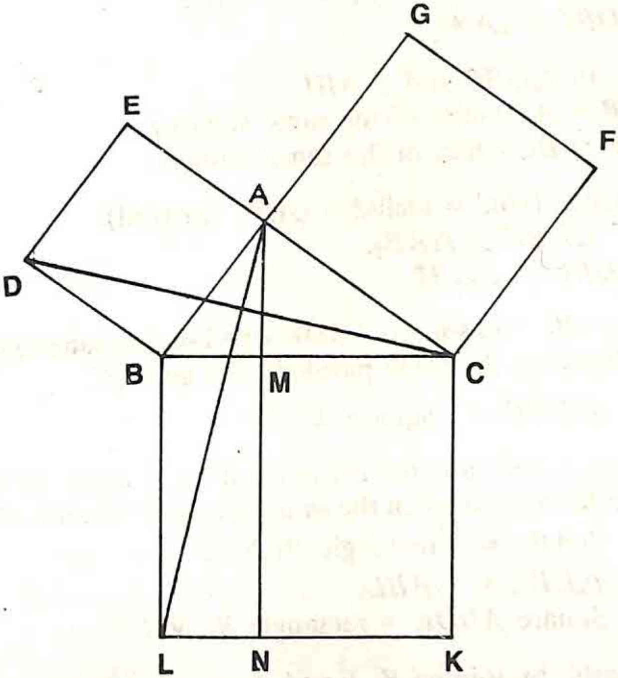


Fig. 198

Let $\triangle ABC$ be a triangle right-angled at A.

To prove that, $BC^2 = AB^2 + AC^2$.

Construction : Squares $BCKL$, $ABDE$ and $ACFG$ are drawn on BC, AB and AC respectively. A straight line parallel to BL is drawn through the point A to interest BC and LK at the points M and N respectively. D, C and A, L are joined.

Proof : $\because$ $BL \parallel AN$,

$\therefore$ AN is perpendicular on BC and LK; that is, each of the quadrilaterals $BLNM$ and $MNKC$ is a rectangle.

$\because$ Each of $\angle BAE$ and $\angle BAC$ is one right angle,

$\therefore$ C, A, E are collinear.

Similarly, B, A, G are collinear.

Again, $\angle DBA = \angle CBL$ (each 90°),

$\therefore \angle DBA + \angle ABC = \angle CBL + \angle ABC$
$= \angle ABC + \angle CBL.$
$\therefore \angle DBC = \angle ABL.$

Now, in $\triangle DBC$ and $\triangle ABL$,
$DB = AB$ (sides of the same square),
$BC = BL$ (sides of the same square),

included $\angle DBC$ = included $\angle ABL$ (proved).
$\therefore \triangle DBC \perp \triangle ABL.$
$\triangle DBC \cong \triangle ABL.$

$\because$ $\triangle DBC$ and square $ABDE$ stand on the same base DB and between the same parallels DB and EC,
$\therefore \triangle DBC = \frac{1}{2}$ square $ABDE$.

Again, $\triangle ABL$ and the rectangle $BNLM$ stand on the same base BL and between the same parallels BL and AN,
$\therefore \triangle ABL = \frac{1}{2}$ rectangle $BLNM$.
$\because \triangle DBC = \triangle ABL.$
$\therefore$ Square $ABDE$ = rectangle $BLNM$.

Similarly, by joining B, F and A, K it can be proved that,

Square $ACFG$ = rectangle $MNKC$.
$\therefore$ Square $ABDE$ + Square $ACFG$
= Rectangle $BLNM$ + Rectangle $MNKC$
= Square $BCKL$.
$\therefore BC^2 = AB^2 + AC^2.$

[*Remarks:* (1) Alternative proof is based on the properties relating to area.

(2) You have verified the algebraic formulae $(a+b)^2 = a^2 + 2ab + b^2$, $(a-b)^2 = a^2 - 2ab + b^2$ and $a^2 - b^2 = (a+b)(a-b)$ geometrically in Class VII. You also know, what the geometrical representations of these formulae are, if a, b represent two line segments. In fact, we have applied these properties in geometrical problems. We can prove the Pythagoras' Theorem by applying the first

formula. The proof is given below:

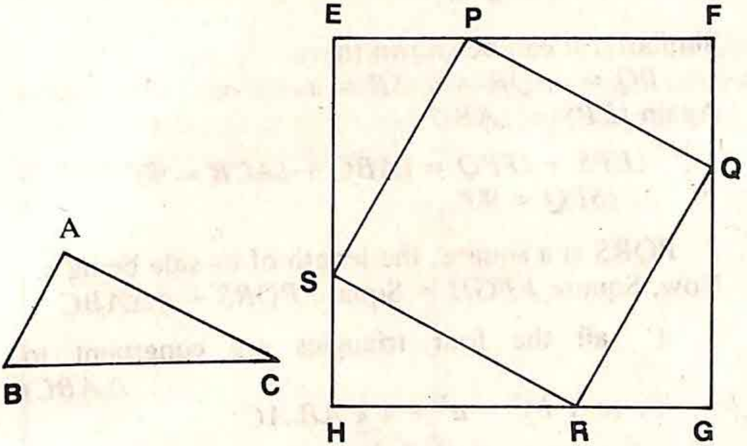


Fig. 199

Let ABC be a triangle, right-angled at A.
To prove that, $BC^2 = AB^2 + AC^2$.

Let us denote the sides BC, CA and AB by a, b, c respectively.
To prove that $a^2 = c^2 + b^2$.

Construction: A square $EFGH$ is drawn, the length of its side being equal to the sum of the lengths of the line segments and b. Four points P, Q, R, S are taken on EF, FG, GH and HE respectively such that $EP = FQ = GR = HS = c$. P, Q; Q, R; R, S; S, P are joined.

Proof: $\because$ $EF = FG = GH = HE = c + b$ (by construction).
$\therefore$ PF = QG = RH = SE = b,
$\therefore$ in $\triangle ABC$ and $\triangle EPS$,
$AB = EP$,
$AC = ES$,
included $\angle BAC$ = included $\angle PES$ (each being a right angle),

$\therefore$ $\triangle ABC \cong \triangle EPS$
$\therefore$ $BC = PS$ (corresponding sides);

but $BC = a$

$\therefore\ PS = a$

Similarly, it can be shown that

$PQ = a,\ QR = a,\ SR = a$

Again $\angle EPS = \angle ABC$.

$\therefore\ \angle EPS + \angle FPQ = \angle ABC + \angle ACB = 90°$

$\therefore\ \angle SPQ = 90°$

$\therefore$ PQRS is a square, the length of its side being a.

Now, Square $EFGH$ = Square $PQRS + 4\triangle ABC$

($\because$ all the four triangles are congruent to $\triangle ABC$)

$$\therefore\ (c + b)^2 = a^2 + 4.\tfrac{1}{2}\, AB.AC$$

$$= a^2 + 2cb$$

$$\therefore\ c^2 + 2cb + b^2 = a^2 + 2cb.$$

$$\therefore\ c^2 + b^2 = a^2.$$

that is, $a^2 = c^2 + b^2$].

Theorem 52

If the area of the square on one side of a triangle be equal to the sum of the areas of the squares on the other two sides, the triangle is right-angled, the angle opposite to the greatest side being right angle.

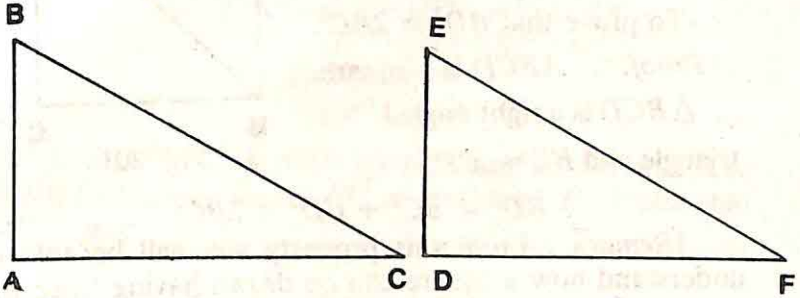


Fig. 200

Let ABC be a triangle, in which $BC^2 = AB^2 + AC^2$. To prove that ABC is a triangle, right-angled at A.

construction: A right-angled triangle DEF is drawn, so that $\angle D = 1$ right angle, $DE = AB$ and $DF = AC$.

Proof: $\because$ $\triangle DEF$ is a right-angled triangle and EF is the hypotenuse,

$\therefore$ $EF^2 = DE^2 + DF^2$

$= AB^2 + AC^2$ ($\because$ $DE = AB$ and $DF = AC$, by construction)

$= BC^2$

$\therefore$ $EF = BC$

$\therefore$ in $\triangle ABC$ and $\triangle DEF$,

$AB = DE$ (by construction),

$AC = DF$ (by construction),

$BC = EF$ (proved).

$\therefore$ $\triangle ABC \cong \triangle DEF$.

$\therefore$ $\angle BAC = \angle EDF$ (corresponding angles)

$= 1$ right angle.

$\therefore$ $\triangle ABC$ is a triangle. right-angled at A

Some Applications:

1. The square drawn on the diagonal of a square is twice the latter.

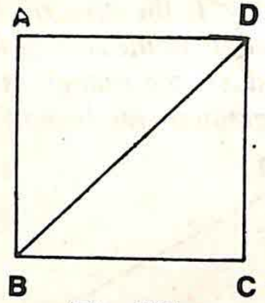


Fig. 201

Let *ABCD* be a square and *BD*, one of its diagonals.

To prove that $BD^2 = 2BC^2$.

Proof: $\because$ *ABCD* is a square.

$\therefore$ $\triangle BCD$ is a right-angled triangle and $BC = CD$.

$$\therefore\ BD^2 = BC^2 + CD^2 = 2BC^2$$

[*Remark* : From this property you will be able to understand how a square can be drawn having twice the area of a given square.]

2. *ABCD* is a rectangle and *O* is any point within it. Prove that, $OA^2 + OC^2 = OB^2 + OD^2$.

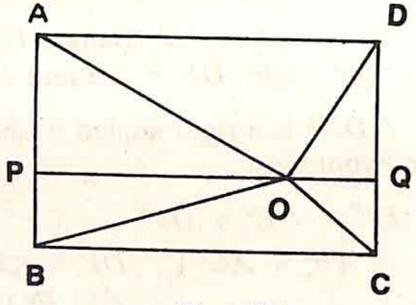


Fig. 202

Construction: Through the point *O*, a straight line parallel to *BC* is drawn to meet *AB* and *DC* at the points *P* and *Q* respectively.

Proof: $\because BC \parallel PQ$

$$\therefore\ PQ \perp AB,\ PQ \perp CD$$

$\therefore$ each of $\triangle APO$, $\triangle BPO$, $\triangle DQO$ and $\triangle CQO$ is right-angled.

$$\therefore\ OA^2 = AP^2 + OP^2,$$
$$OB^2 = BP^2 + OP^2,$$
$$OC^2 = CQ^2 + OQ^2,$$

$$OD^2 = DQ^2 + OQ^2.$$

$$\therefore\ OA^2 + OC^2 = AP^2 + OP^2 + CQ^2 + OQ^2.$$

But according to construction, both the quadrilaterals *APQD* and *BPQC* are rectangles.

$$\therefore\ AP = DQ \text{ and } CQ = BP.$$

$$\begin{aligned} OA^2+OC^2 &= DQ^2+OP^2+BP^2+OQ^2 \\ &= DQ^2+OQ^2+BP^2+OP^2 \\ &= OD^2+OB^2 \\ &= OB^2+OD^2 \end{aligned}$$

3. $\triangle ABC$ is an isosceles right-angled triangle. The bisector of $\angle BAC$ meets *BC* at the point *D*. Prove that $CD^2=2BD^2$.

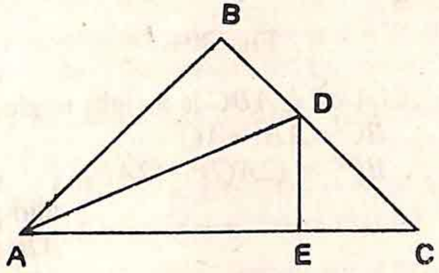


Fig. 203

Construction : DE is drawn perpendicular on *AC* from the point *D*.

Proof : $\because$ $\triangle ABC$ is an isosceles right-angled triangle,

$$\therefore\ \angle ABC = 45^\circ.$$

$\therefore$ $\triangle DEC$ is an isosceles right-angled triangle.

Again, in $\triangle ABC$ and $\triangle AED$.

$\angle BAD = \angle EAD$ (by hypothesis),

$\angle ABD = \angle AED$ (both right angles),

side *AD* is common (corresponding sides).

$$\therefore\ \triangle ABD \cong \triangle AED.$$

$$\therefore\ BD = DE.$$

Now, $\because$ $\triangle DEC$ is an isosceles right-angled triangle,

$$\therefore\ DC^2 = 2DE^2$$
$$= 2BD^2.$$

4. ABC is a triangle right-angled at A. BP and CQ are two medians. Prove that $5BC^2=4(BP^2 + CQ^2)$.

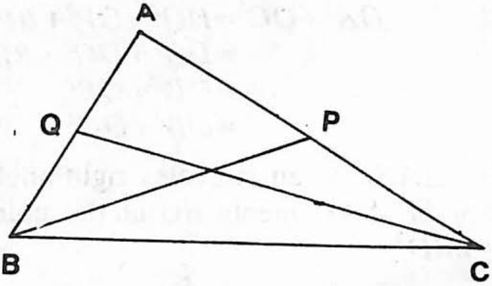


Fig. 204

Proof : $\because$ $\angle A$ of $\triangle ABC$ is a right angle,

$\therefore\ BC^2=AB^2+AC^2$

$\therefore\ BC^2 = (2AQ)^2+(2AP)^2$ ($\because$ Q and P are mid-points of AB and AC respectively).

$\therefore\ BC^2 = 4(AQ^2+AP^2)$.

Again, $\triangle BAP$, $\triangle CAQ$ are right-angled triangles,

$$\therefore\ BP^2 = AB^2 + AP^2$$
$$= (2AQ)^2 + AP^2$$
$$= 4AQ^2 + AP^2$$

and $CQ^2=AC^2+AQ^2$

$$=(2AP)^2+AQ^2$$
$$=4AP^2+AQ^2$$

$$\therefore\ BP^2+CQ^2 =4AQ^2+AP^2+4AP^2+AQ^2$$
$$=5(AQ^2+AP^2).$$

$$5BC^2=5.4(AQ^2 + AP^2)$$
$$=4.5(AQ^2+AP^2)$$
$$=4(BP^2+CQ^2).$$

Exercise 27

1. PQR is a triangle, right-angled at Q. S is any point on QR. Prove that $PS^2+QR^2=PR^2+QS^2$.

2. Prove that the sum of the squares on the sides of a rhombus is equal to the sum of the squares on its diagonals.

3. ABC is an equilateral triangle. AD is perpendicular to BC. Prove that $AB^2+BC^2+CA^2=4AD^2$.

4. ABC is a triangle, right-angled at A. P and Q are two points on AB and AC respectively. P, Q; B, Q and C, P are joined. Prove that $BQ^2+PC^2=BC^2+PQ^2$.

5. From a point Q within a triangle ABC, OX, OY, OZ are drawn perpendiculars on BC, CA and AB respectively. Prove that $AZ^2+BX^2+CY^2=AY^2+CX^2+BZ^2$.

6. The diagonals of the quadrilateral $ABCD$ intersect each other at right angles. Prove that $AB^2+CD^2=BC^2+DA^2$.

7. AD is the altitude of $\triangle ABC$. If $AB>AC$, prove that $AB^2-AC^2=BD^2-CD^2$.

8. RST is a triangle, right-angled at S, X and Y are mid-points of RS and ST respectively. Prove that $RY^2+XT^2=5XY^2$.

9. P is any point outside the square $ABCD$. If $PA>PB$, Prove that $PA^2-PB^2=PD^2-PC^2$.

10. The line segment AB is produced to the point C, so that $AC=3AB$. An equilateral triangle BCD is drawn with BC as a side. Prove that $AD^2=7AB^2$.

11. $ABCD$ is a square. The points E, F, G, H are taken on AB, BC, CD, DA respectively, so that $AE=BF=CG=DH$. Prove that $FH^2=2(AE^2+BE^2)$.

12. ABC is a triangle. The perpendiculars drawn from the vertices B and C on the sides AC and AB respectively intersect each other at the point P. Prove that $AC^2+BP^2=AB^2+CP^2$.

13. $\triangle ABC$ is an isosceles right-angled triangle with $\angle C$ as right angle. D is any point on AB. Prove that $AB^2+DB^2=2CD^2$.

14. ABC is a triangle, right-angled at A. CD is a median. Prove that $BC^2=CD^2+3AD^2$.

CHAPTER 6

CONSTRUCTIONS

6.1 Constructions of circum-circle

CONSTRUCTION 5

To draw a circle circumscribing a given triangle.

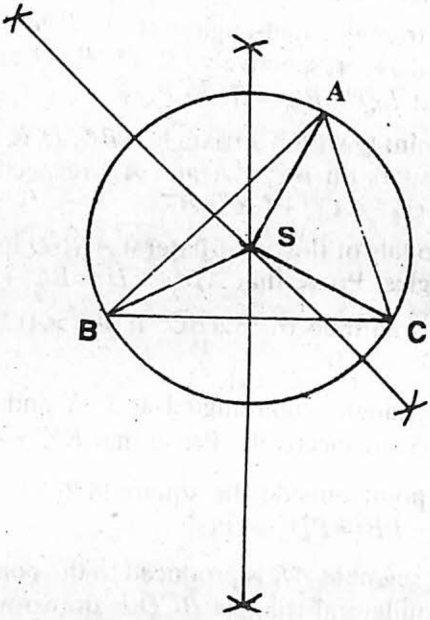


Fig. 205

Let ABC be a triangle.

To draw a circle circumscribing $\triangle ABC$.

Construction : Perpendicular-bisectors of sides AB and BC are drawn to meet at the point S. With S as centre and SA *(or* SB, or SC) as radius a circle is drawn.

This is the circle that circumscribes $\triangle ABC$.

Proof : $S, A; S, B; S, C$ are joined.

$\because$ S, is a point on the perpendicular-bisector of AB,

$\therefore$ S is equidistant from A and B
that is, $SA=SB$.

Similarly, $SB=SC$.
$\therefore$ $SA=SB=SC$.

$\therefore$ the circle drawn with S as centre and SA as radius, passes through the points B and C, that is, the circle passes through three vertices A, B and C of $\triangle ABC$.
$\therefore$ this circle is the circum-circle.

[*Remark* : (1) Analysis of the method of construction : From the definition of a circum-circle it follows that the sides of the triangle are chords of the circle. We also know that the perpendicular-bisector of a chord passes through the centre. Therefore, the point of intersection of the perpendicular-bisectors (of two sides) is the centre of the circle. We have also come to know that the distances of the vertices of a triangle from the point of intersection of the perpendicular-bisectors of the sides are equal. Therefore, the distance of any vertex from the point of intersection is the length of the radius of the circle.

(2) The circum-centre lies within the triangle, on the hypotenuse or outside the triangle on the opposite side of the greatest side according as the triangle is acute-angled, right-angled or obtuse-angled.

(3) The mid-point of the hypotenuse of a right-angled triangle is the circum-centre.

(4) It is interesting to note that circum-centre is obtained by drawing an isosceles triangle. Draw an isosceles triangle with any side of the given triangle as base. Take the base angles equal to complement of the angle opposite to that side. The vertex of that isosceles triangle is the circum-centre and one of the equal sides of that triangle is the radius. For example, if with BC as base

two angles equal to (90° − $\angle A$) are drawn at the points B and C, then also the point S is obtained.]

An Application :

1. In $\triangle ABC$, $AB = 4{\cdot}5$ cm, $\angle BAC = 60°$, $\angle ABC = 75°$; draw the triangle. Again in $\triangle ABD$, $\angle BAD = 90°$, $\angle ABD = 45°$ and the point D will be on that side of AB in which the point C lies; draw $\triangle ABD$. Draw the circle circumscribing $\triangle ABD$. Observe, whether the point C lies on the circum-circle. Justify your observation.

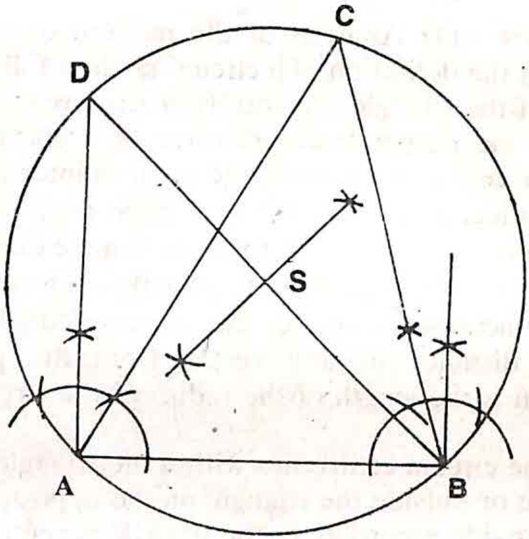


Fig. 206

Construction : A line segment AB=4·5 cm is drawn with the help of a ruler. An angle of 60° is drawn at the point A with AB as side and an angle of 75° is drawn at the point B with BA as side. The other two arms of these angles intersect at the point C.

Thus, $\triangle ABC$ is drawn.

Again, an angle of 90° is drawn at the point A with AB as side and an angle of 45° is drawn at the point B with BA

as side. The angles are so drawn that the other two arms of these angles intersect at the point D, which lies on that side of AB in which the point C lies.

Hence, $\triangle ABD$ is drawn.

Now the mid-point S of the side BD is determined; a circle is drawn with S as centre and SD as radius.

This is the circle that circumscribes $\triangle ABD$.

It is observed that the point C lies on the circum-circle.

Proof: According to construction, $\triangle ABC$ and $\triangle ABD$ are two required triangles.

$\because \angle BAD=90^\circ$,

$\triangle BAD$ is a right-angled triangle with BD as hypotenuse.

$\because$ the mid-point of the hypotenuse is the centre of the circle circumscribing the right-angled triangle and the length of the radius is half the length of the hypotenuse;

$\therefore$ the circle drawn with S as centre and SD as radius is the circum-circle of $\triangle ABD$.

$\because \angle BAD=90^\circ, \angle ABD=45^\circ$,

$\therefore \angle ADB=45^\circ$

Again, $\because \angle BAC=60^\circ, \angle ABC=75^\circ$,

$\therefore \angle ACB=45^\circ$,

$\therefore \angle ADB=\angle ACB$.

$\therefore$ the four points A, B, C, D are concyclic.

the point C lies on the circle circumscribing $\triangle ABD$.

Exercise 28

[State the method of construction and give proof in each case.]

1. Draw the following triangles and draw the circum-circle of each triangle. Find the length of the radius of the circle in each case with the help of a scale.

(i) The lengths of three sides are 4 cm, 7 cm, 9 cm.

(ii) The lengths of three sides are 6 cm, 8 cm, 10 cm.

(iii) The lenghts of three sides are 5 cm, 7 cm, 8 cm.

(iv) The lengths of two sides are 4·5 cm and 6·5 cm and the angle included by them is 60°.

(v) The length of a side is 8·2 cm and angles adjacent to that side are 30° and 45°,

(vi) The triangle is equilateral, having each side equal to 6 cm.

(vii) The triangle is isosceles, with the length of the base being equal to 7·2 cm and each equal side 9 cm.

(viii) The triangle is isosceles, with the length of the base as 5·6 cm and each base angle as 75°.

(ix) The triangle is right-angled, with two sides containing the right angle equal to 4 cm and 7 cm.

(x) The triangle is right-angled with hypotenuse as 12 cm and another side as 5 cm.

2. Given $PQ=7{\cdot}5$ cm, $\angle QPR=50°$, $\angle PQR=70°$,
$PO=7{\cdot}5$ cm, $\angle QPS=40°$, $\angle PQS=80°$;

draw $\triangle PQR$ and $\triangle PQS$ such that the points R and S lie on the same side of PQ. Draw the circle circumscribing $\triangle PQR$. Observe the position of the point S. Is it within the circle, on the circle or outside the circle?

3. Given, $AB=4{\cdot}8$ cm, $\angle BAC=55°$, $\angle ABC=60°$,
$AB=4{\cdot}8$ cm, $\angle BAD=35°$, $\angle ABD=30°$,

draw $\triangle ABC$ and $\triangle ABD$ such that the points C and D lie on opposite side of AB. Draw the circle circumscribing $\triangle ABC$. State what other charcteristic you observe.

4. Draw the following triangles. Draw the circum-circle of each triangle. Find the lengths of their radii with the help of a

scale and state the relation, if any, between the radii.

(i) In $\triangle ABC$, $AB=3{\cdot}9$ cm, $\angle BAC=40°$, $\angle ABC=60°$;
(ii) In $\triangle PQR$, $PQ=3{\cdot}9$ cm, $\angle QPR=45°$, $\angle PQR=55°$;
(iii) In $\triangle XYZ$, $XY=3{\cdot}9$ cm, $\angle YXZ=30°$, $\angle XYZ=70°$;
(iv) In $\triangle DEF$, $DE=3{\cdot}9$ cm, $\angle EDF=50°$, $\angle DEF=50°$;

5. Draw the quadrilateral $ABCD$, having $AB=4$ cm, $BC=7$ cm, $CD=4$ cm, $\angle ABC=60°$, $\angle BCD=60°$. Draw the circle circumscribing $\triangle ADC$. State what other characteristic you observe.

6. Draw the rectangle $PQRS$, having $PQ=4$ cm and $QR=6$ cm. Draw the diagonals of the rectangle. State where the centre of the circum-circle of $\triangle PQR$ lies and how much the length of the radius is. Draw the circum-circle.

6.2 Construction of in-circle :

CONSTRUCTION 6

To draw a circle inscribed in a given triangle.

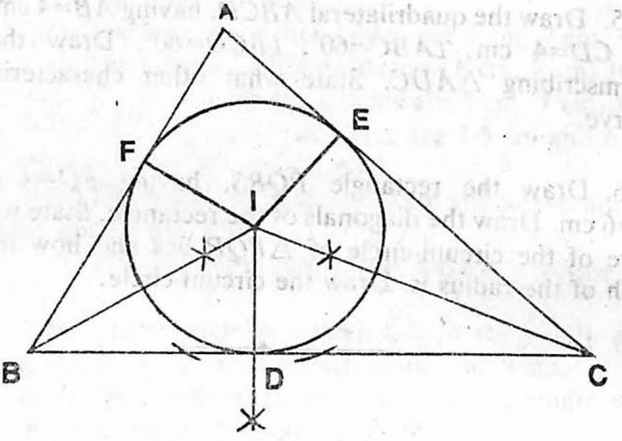


Fig. 207

Let ABC be a triangle.
To draw the circle inscribed in $\triangle ABC$.

Construction : $\angle ABC$ and $\angle ACB$ are bisected. The bisectors meet at the point I. From the point I perpendicular to the side BC is drawn to meet BC at the point D. A circle is drawn with centre I and radius ID. That is the circle inscribed in $\triangle ABC$.

Proof : IE and IF are perpendiculars to AC and AB respectively.

$\because$ the point I is on the bisector of $\angle ABC$,
$\therefore$ the point I is equidistant from BC and BA.
$\therefore$ $ID=IF$
Similarly, $ID=IE$. $\therefore$ $ID=IE=IF$.

$\therefore$ the circle drawn with centre I and radius ID passes through the points E and F.

Again, ∵ ID, IE, IF are perpendiculars to BC, CA and AB respectively,

∴ BC, CA and AB are tangents to the circle.

∴ the circle is inscribed in $\triangle ABC$.

[*Remarks :* Analysis of the method of construction : From the definition of the in-circle, we know the sides of the triangle are tangents to the in-circle. We also know that the bisector of the angle included by two tangents passes through the centre, that is, the bisector of any angle of a triangle passes through the in-centre. Therefore, the point of intersection of the bisectors of two angles of triangle is the in-centre. Again, since the sides of the triangle are tangents to the in-circle, the perpendicular length of any side from the centre is the radius. In this way we can obtain the centre and radius of the in-circle and hence the circle can be drawn.

(2) In case of an equilateral triangle, the bisector of an angle and perpendicular bisector of the opposite side are the same straight line; so the circum-centre and the in-centre of an equilateral triangle are the same point.]

Exercise 29

[State the method of construction and give proof in each case]

1. Draw the following triangles and draw the in-circle of each triangle. Find the length of the radius of the circle with the help of a scale in each case.

(i) The lengths of three sides are 8·5 cm, 6·5 cm and 5 cm.

(ii) The lengths of two sides are 7·3 cm, 6 cm and the angle included by those two sides is 75°.

(iii) The length of a side is 6·9 cm and measures of two angles adjacent to this side are 60° and 75°.

(iv) The triangle is right-angled, with two sides containing the right angle equal to 7 cm and 9 cm.

(v) The triangle is right-angled, with hypotenuse as 10 cm and another side as 6·5 cm.

(vi) The triangle is isosceles, with the length of the base as 8·3 cm and each of the equal sides 7·1 cm.

(vii) The triangle is isosceles, with the length of the base as 10 cm and each of the equal angles 45°.

2. Draw an equilateral triangle having each side equal to 7 cm. Draw the circum-circle and in-circle of that triangle. Find the circum-radius and in-radius with the help of a scale. Observe whether there is any relation between them.

6.3 Construction of tangents :

CONSTRUCTION 7

To draw a tangent to a circle at a point on it.

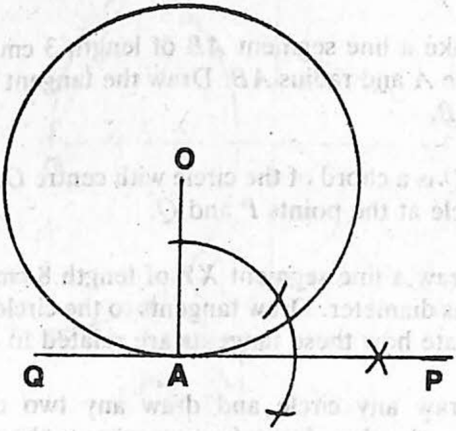


Fig. 208

Let A be a point on a circle with centre O.
To draw a tangent to the circle at the point A.

Construction : O, A are joined. PQ is drawn perpendicular to OA at the point A.
Then PQ is the required tangent.

Proof : We know that the perpendicular to any radius of circle at the point where that radius meets the circle is the tangent to the circle at that point.
$\because$ OA is radius of the circle
and $OA \perp PQ$ (by construction)
$\therefore$ PQ is tangent to the circle at the point A.

Exercise 30

[State the method of construction and give proof in each case.]

1. Draw a circle of 5 cm radius. Draw a tangent to that circle at any point on it.

2. Take a line segment AB of length 3 cm. Draw a circle with centre A and radius AB. Draw the tangent to that circle at the point B.

3. PQ is a chord of the circle with centre O. Draw tangents to the circle at the points P and Q.

4. Draw a line segment XY of length 8 cm. Draw a circle with XY as diameter. Draw tangents to the circle at the points X and Y. State how these tangents are related to each other.

5. Draw any circle and draw any two diameters right-angled to each other. Draw four tangents to the circle at the four end-points of the two diameters. State what kind of quadrilateral is formed as a result.

6. Draw an equilateral triangle ABC, having each side of length 6 cm. Draw the circle circumscribing $\triangle ABC$. Draw tangents to the circum-circle at the points A, B and C.

CONSTRUCTION 8

To draw a tangent to a circle from a point outside it.

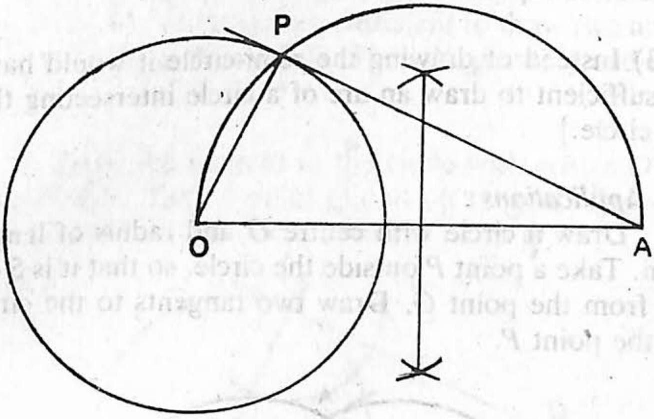


Fig. 209

Let A be a point outside a circle with centre O.
To draw a tangent to the circle from the point A.

Construction : O, A are joined. A semi-circle is drawn with OA as diameter to meet the circle at the point P. A, P are joined and AP is produced.
Then AP is the required tangent.

Proof : O, P are joined.
$\because \angle OPA$ is an angle in the semi-circle,
$\therefore \angle OPA = 1$ right angle.
$\therefore PA \perp OP$.
$\therefore PA$ is tangent to the circle at the point P.

[*Remarks* : (1) Analysis of method of construction : Let us suppose that AP is a tangent. Then $\angle OPA$ is a right angle. To get this right angle we are to draw a semi-circle, whose diameter is OA and P is the point of intersection of this semi-circle and the given circle.

(2) If instead of a semi-circle, a circle is drawn with *OA* as diameter, it would have intersected the given circle at another point. Therefore, two tangents can be drawn to a circle from a point outside it.

(3) Instead of drawing the semi-circle it would have been sufficient to draw an arc of a circle intersecting the given circle.]

Some Applications :

1. Draw a circle with centre *O* and radius of length 2·5 cm. Take a point *P* outside the circle, so that it is 5 cm apart from the point *O*. Draw two tangents to the circle from the point *P*.

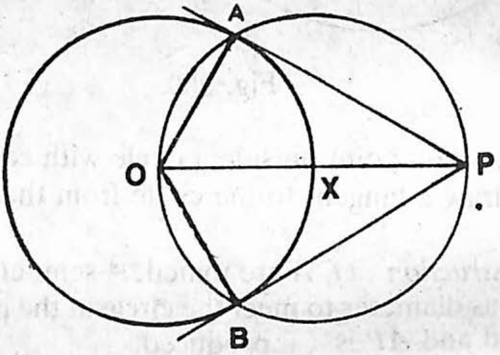


Fig. 210

Construction : A line segment *OX* of length 2·5 cm is taken. A circle is drawn with centre *O* and radius *OX*. Now, *OX* is produced to the point *P*, such that *XP*=*OX*.

Again, a circle is drawn with centre *X* and radius *OX*, which intersects the circle with centre O at the points A and B. *P*, *A* and *P*, *B* are joined and produced.

Then *PA* and *PB* are the required two tangents.

Proof : ∵ $OX = 2{\cdot}5$ cm,

∴ *AXB* is the required circle.

Again, ∵ $OX = XP$,

∴ *OP* is the diameter of the circle with centre *X*.

$\therefore \angle OAP = \angle OBP = 1$ right-angle.

$\therefore$ PA and PB are the tangents to that circle.

[*Remark* : Instead of drawing the complete circle with centre X it would have been sufficient to draw two arcs to meet the circle with centre O. Then the points A and B are obtained.]

2. Draw the tangent to the circle with centre O at a point P on it. Take a point Q on that tangent. Draw the other tangent to that circle from the point Q.

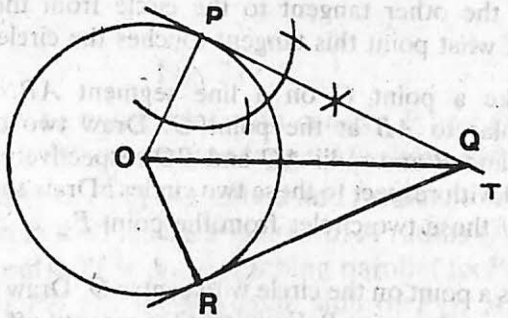


Fig. 211

Construction : O, P are joined. PT is drawn perpendicular to OP at the point P.

PT is the tangent to that circle at the point P.

A point Q is taken on PT. An arc of a circle is drawn with centre Q and radius QP to meet the circle with centre O at the point R. Q, R are joined.

Then QR is the other tangent to that circle from the point Q.

Proof : O, R; O, Q are joined.

In $\triangle ORQ$ and $\triangle OPQ$,

$OR=OP$ (radius of the same circle),

$RQ=PQ$ (by construction),

OQ is the common side;

$\therefore \triangle ORQ \cong \triangle OPQ$

$\therefore \angle ORQ=\angle OPQ=1$ right angle.

$\therefore$ QR is the tangent to the circle at the point R.

Exercise 31

[State the method of construction and give proof in each case.]

1. Draw a circle with radius 2·5 cm. Take a point outside that circle 6·5 cm apart from the centre. Draw a tangent to that circle from that exterior point. Find the length of that tangent with the help of a scale.

2. Draw an equilateral triangle having side 5 cm. Draw a circle circumscribing that triangle. Draw a tangent to that circle at the point A. Take a point P on that tangent such that $AP=5$ cm. Draw the other tangent to the circle from the point P. Observe at what point this tangent touches the circle.

3. Take a point O on a line segment AB. Draw PQ perpendicular to AB at the point O. Draw two circles with centres A and B and radii AO and BO respectively. What can PQ be said with respect to those two circles? Draw the other two tangents of those two circles from the point P.

4. P is a point on the circle with centre O. Draw the tangent to the circle at the point P. From that tangent cut off PQ making equal to the length of the radius of the circle. Draw QR, the other tangent to the circle from the point Q. Find the measure of $\angle PQR$ with the help of a protractor.

CONSTRUCTION 9

To draw a direct common tangent to two circles.

First Method:

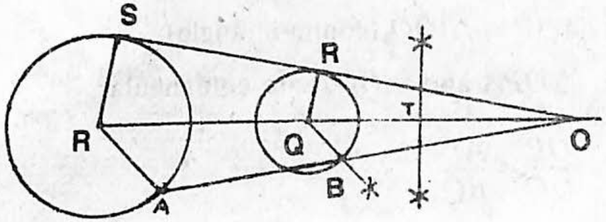


Fig. 212

Let there be two circles with centres P and Q.

To draw a direct common tangent to those two circles.

Construction: P, Q are joined and produced..A radius PA of the circle with centre P is drawn. A radius QB of the circle with centre Q is drawn making parallel to PA, such that PA and PB are on the same side of PQ. A, B are joined and AB is produced. AB produced meets PQ produced at the point O. The mid-point T of OQ is obtained. An arc of a circle with centre T and radius TQ is drawn to meet the circle with centre Q at the point R. O, R are joined and OR is produced.

Then the straight line OR is the required direct common tangent.

Proof: Q, R are joined and the perpendicular to the straight line OR is drawn from the point P to meet the straight line OR at the point S.

According to construction, OR is tangent to the circle with centre Q.

Again, $PA \parallel PB$ and AB is the transversal,

$\therefore \angle PAB = \angle QBO$ (corresponding angles).

Similarly, $\angle APO = \angle BQO$ (corresponding angles)

[$\because$ $PA \parallel QB$ and PQ is the transversal]

$\therefore$ in $\triangle OPA$ and $\triangle OQB$,

$\angle PAO = \angle QBO$,
$\angle APO = \angle BQO$.

and $\angle AOP = \angle BOQ$ (common angle),

$\therefore$ $\triangle OPA$ and $\triangle OQB$ are equiangular.

$$\therefore \frac{OP}{OQ} = \frac{AP}{BQ}$$

Again, PS and QR are perpendiculars to the straight line OR,

$\therefore$ $PS \parallel QR$ and SR and PQ are the transversals.

$\therefore$ in $\triangle OPS$ and $\triangle OQR$,

$\angle PSO = \angle QRO$ (corresponding angles),

$\angle SPO = \angle RQO$ (corresponding angles),

$\angle POS = \angle QOR$ (corresponding angles);

$\therefore$ $\triangle OPS$ and $\triangle OQR$ are equiangular.

$$\therefore \frac{OP}{OQ} = \frac{PS}{QR}$$

$$\therefore \frac{PS}{QR} = \frac{AP}{BQ} = \frac{AP}{QR}$$ [$BQ = QR$, $\because$ they are radii of the circle with centre Q].

$\therefore$ $PS = AP =$ radius of the circle with centre P and $\angle PSR = 1$ right angle.

$\therefore$ OS is the tangent to the circle with centre P.

$\therefore$ OR, that is, ORS is the common tangent to two circles since, the circles are on the same side of OR; therefore, OR is direct common tangent.

Second Method:

Let there be two circles with centres P and Q. To draw a direct common tangent to these two circles.

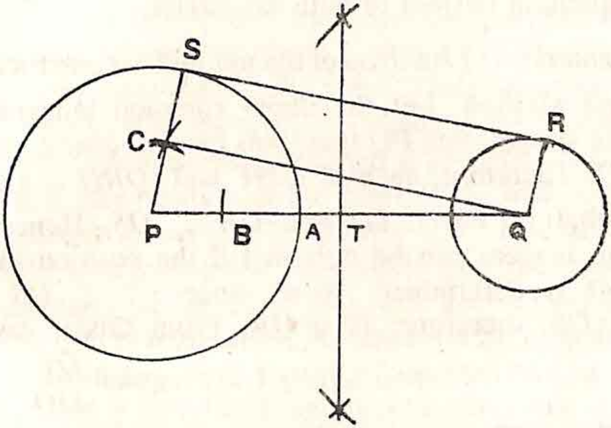


Fig. 213

Construction: P, Q are joined and T, the mid-point of PQ, is determined. Let PQ meet the circle with centre P at the point A. From AP, AB is cut off making equal to the radius of the circle with centre Q. Two arcs are drawn, one with centre T and radius TP and the other with centre P and radius PB. Let these arcs meet each other at the point C. P, C are joined and PC is produced to meet the circle at the point S. C, Q are joined. An arc of a circle is drawn with the point S as centre and CQ as radius to meet the circle with centre Q at the point R. S, R are joined.

Then SR is the required direct common tangent.

Proof: According to construction, $\angle PCQ$ is the angle in the semi-circle.

$\therefore \angle PCQ = 1$ right angle.

Again, $CS = PS - PC = PA - PB = AB = QR$ and $SR = CQ$ (by construction),

$\therefore$ $\underline{CQRS}$ is a parallelogram of which $\angle QCS = 1$ right angle.

$\therefore$ $CQRS$ is a rectangle.

$\therefore$ each of $\angle PSR$ and $\angle QRS$ is 1 right angle.

$\therefore$ $SR \perp PS$ and $SR \perp QR$

$\therefore$ SR is tangent to both the circles, that is, SR is the direct common tangent to both the circles.

[*Remarks:* (1) Analysis of the method of construction:

First Method: Let the direct common tangent SR meet the straight line PQ that joins the two centres at the point O. Therefore, each of $\angle OSP$ and $\angle ORQ$ is a right angle, that is, $PS \perp OS$ and $QR \perp OS$. Hence the common tangent can be obtained if the position of the point O is determined. Now, since $PS \perp OS$ and $QR \perp OS$, therefore, $PS \parallel QR$. From this it can be *proved that* $\triangle OPS$ and $\triangle OQR$ are equian-$\frac{OP}{OQ}=\frac{PS}{QR}$, gular; therefore, that is, the line segment PQ is divided externallly at the point O in the ratio of the two radii. So two parallel radii PA and QB are drawn and the line segment that joins their end-points is produced to meet PQ produced. Thus the position of the point O is obtained.

Second Method:

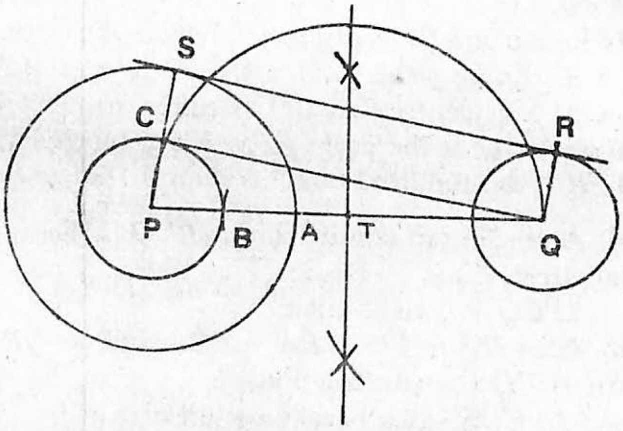


Fig. 214

Let SR be the direct common tangent; therefore, $PS \perp SR$ and $QR \perp SR$. $\therefore$ $PS \parallel QR$. $RS \parallel QC$ is drawn.

Therefore, *CQRS* is a rectangle. Now, if the position of the point *C* is found out, the positions of the points *S* and *R*, that is the common tangent *SR* is obtained. From the rectangle *CQRS,* we get *QR* = *CS*. Therefore, *PC* being the difference of two radii and ∠*PCQ* being a right angle, it is an angle in the semi-circle drawn on *PQ* as diameter. Therefore, the point *C* is the point of intersection of a new circle drawn with centre *P* and radius equal to difference of two radii and above mentioned semi-circle. For this, to determine the position of the point *C*, the new circle, and the semi-circle are to be drawn. To avoid untidiness instead of drawing the new circle and the semi-circle completely , in fig. 213 , two arcs are drawn. If only construction is required, C, Q need not be joined.

(2) Another direct common tangent could have been drawn; that is two direct common tangents can be drawn to two circles.

(3) For drawing direct common tangents, the two circles are either to be external to each other or intersecting.

(4) This method of construction does not apply when two circles have equal radii. The method of construction to be followed then is given below:

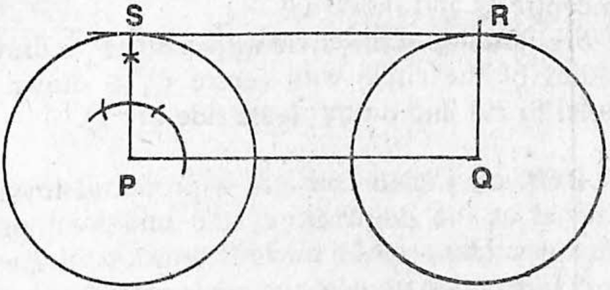


Fig. 215

Let there be two circles of equal radii with centres *P* and *Q*.

To draw a direct common tangent to these two circles.

Construction: P, Q are joined. Perpendicular is drawn to PQ at the point P to meet the circle with centre P at the point S. An arc of a circle is drawn with centre S and radius equal to PQ to meet the circle with centre Q at the point R. S, R are joined.

Then SR is a direct common tangent to two circles.]

Some Applications:

1. Draw two circles of radii 3·5 cm and 2 cm, their centres being 8 cm apart. Draw two direct common tangents to those two circles.

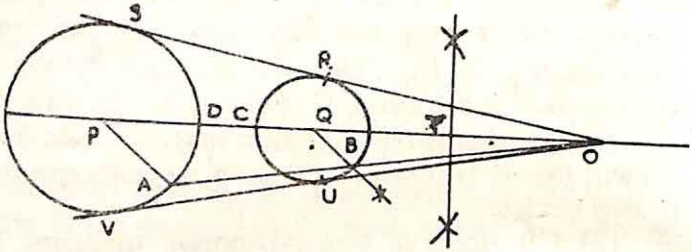


Fig. 216

Construction: PQ is a line segment of length 8 cm. from PQ line segment PD of length 3·5 cm and a line segment QC of length 2 cm are cut off. Two circles are drawn, one with centre P and radius PD and the other with centre Q and radius QC.

PA, a radius of the circle with centre P, is drawn. QB, a radius of the circle with centre Q, is drawn making parallel to PA and on the same side of PQ.

A, B are joined and AB is produced to meet PQ produced at the point O. T, the mid-point of QO is determined. An arc of a circle is drawn with T as centre and TQ as radius to meet the circle with centre Q at the point R. O, R are joined and OR is produced which touches the circle with centre P at the point S. Then OS is a direct common tangent to two circles. Again, an arc of a circle is drawn with centre OR to meet the circle with centre Q at the point U. O, U are joined and OU is produced, which touches the circle with centre P at the point V. Then OV is the second direct common tangent to these two circles.

(Justify the method of construction.)

2. Draw an equilateral triangle, having side 5 cm. Draw circles of radii 2·5 cm with centres A, B and C. Draw the direct common tangents to each pair of these circles, which are exterior to the triangle.

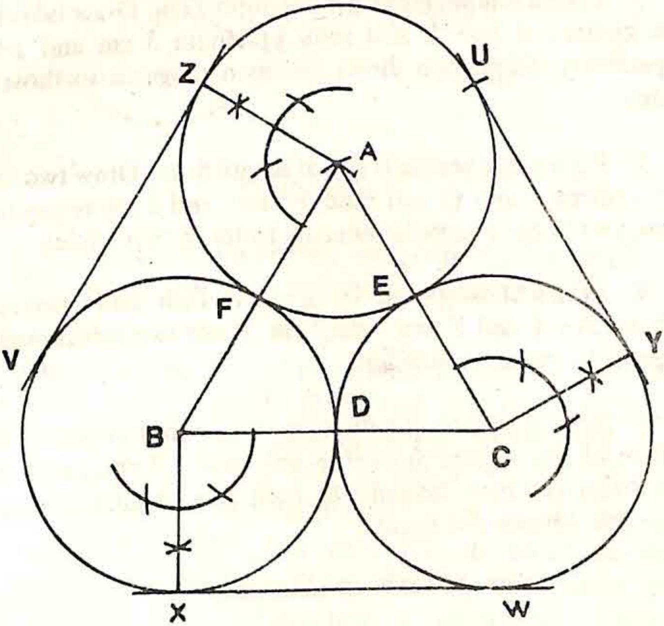


Fig. 217

Construction: The equilateral triangle ABC of side 5 cm is drawn. The portions AF, BD and CF are cut off from AB, BC and CA respectively making equal to 2·5 cm. Three circles are drawn with centres A, B, C and radii AF, BD and CE respectively. Perpendiculars BX, CY and AZ are drawn to BC, CA and AB at the points B, C and A respectively. They meet the three circles at the points X, Y and Z respectively. The circles are drawn with centres X, Y and Z and radii 5 cm to meet the three circles with centre C, centre A and centre B at the points W, U and V

respectively. *X*, *W*; *Y*, *U* and *Z*, *V* are joined.

Then *XW*, *YU* and *ZV* are the required three direct common tangents.

Exercise 32

1. Take a line segment *AB* of length 7 cm. Draw two circles with centres *A* and *B* and radii of length 3 cm and 1·5 cm respectively. Draw two direct common tangents to those two circles.

2. Take a line segment *PQ* of length 5 cm. Draw two circles with centres *P* and *Q* and radii 3·5 cm and 2 cm respectively. Draw two direct common tangents to those two circles.

3. Take a line segment *XY* of length 4 cm. Draw two circles with centres *X* and *Y* and radii 2 cm. Draw two direct common tangents to those two circles.

4. Draw a square of side 3 cm. Draw four circles with the vertices of that square as centres and radii 1·5 cm. Draw those four direct common tangents to each pair of adjacent circles, which are outside the square.

CONSTRUCTION 10

To draw a transverse common tangent to two circles.

First Method:

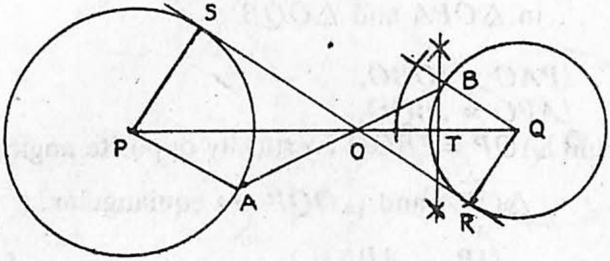


Fig. 218

Let there be two circles with centres P and Q.

To draw a transverse common tangent to two circles.

Construction: P, Q are joined. Draw PA, a radius of the circle, with centre P. Draw QB, a radius of the circle with centre Q making parallel to PA on that side of PQ which is opposite to the side in which PA lies. A, B are joined. AB and PQ intersect each other at the point O. T, the mid-point of OQ is determined. Draw an arc of a circle with centre T and radius TQ to meet the circle with centre Q at the point R. O, R are joined and RO is produced.

Then the straight line OR is the required transverse common tangent.

Proof: Q, R are joined and perpendicular is drawn to the straight line OR from the point P to meet the straight line OR at the point S.

According to construction, OR is tangent to the circle with centre Q.

Again, $PA \parallel QB$ and AB is the transversal,

$\therefore \angle PAO = \angle QBO$ (alternate angles).

Similarly, $\angle APO = \angle BQO$ (alternate angles).
[$PA \parallel QB$ and PQ is the transversal]

$\therefore$ in $\triangle OPA$ and $\triangle OQB$,

$\angle PAO = \angle QBO$,
$\angle APO = \angle BQO$,
and $\angle AOP = \angle BOQ$ (vertically opposite angles);

$\therefore$ $\triangle OPA$ and $\triangle OQB$ are equiangular.

$$\therefore \frac{OP}{OQ} = \frac{AP}{BQ}.$$

Again, PS and QR are perpendiculars to the straight line OR;
$\therefore$ $PS \parallel QR$ and, SR and PQ are transversals.
$\therefore$ in $\triangle OPS$ and $\triangle OQR$,

$\angle PSO = \angle QRO$ (alternate angles),
$\angle SPO = \angle RQO$ (alternate angles),
$\angle POS = \angle QOR$ (vertically opposite angles);

$\therefore$ $\triangle OPS$ and $\triangle OQR$ are equiangular.

$$\therefore \frac{PO}{OQ} = \frac{PS}{QR}.$$

$$\therefore \frac{PS}{QR} = \frac{AP}{BQ} = \frac{AP}{QR}$$ [$BQ = QR$, $\because$ they are radii of the circle with centre Q]

$PS = AP$ = radius of the circle with centre P.
and $\angle PSR = 1$ right angle.

$\therefore$ OS is tangent to the circle with centre P.

$\therefore$ OR, that is, ROS is a transverse common tangent to two circles.

Second Method:

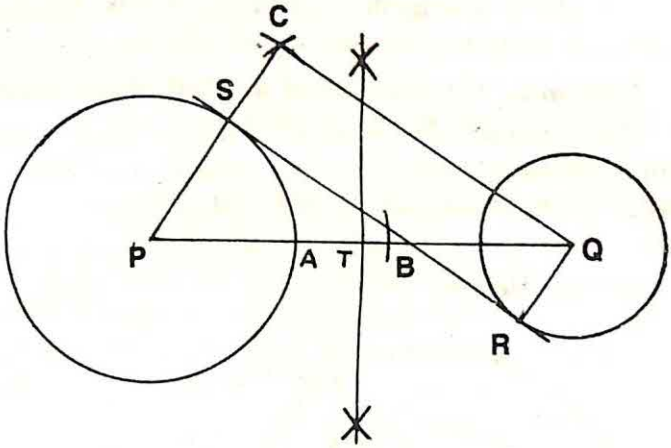


fig. 219

Let there be two circles with centres P and Q.

To draw a transverse common tangent to these two circles.

Construction: P, Q are joined and T, the mid-point of PQ is determined. Let PQintersectthe circle with centre P at the point A. From AQ a portion AB is cut off making equal to the radius of the circle with centre Q. Two arcs are drawn, one with centre T and radius TP and the other with centre P and radius PB. Let these two arcs meet each other at the point C. P, C are joined to meet the circle with centre P at the point S. C, Q are joined. An arc of a circle with centre S and radius equal to CQ is drawn to meet the circle with centre Q at the point R.

Then SR is the required transverse common tangent.

Proof: According to constructions, $\angle PCQ$ is an angle in the semi-circle;

$\therefore \angle PCQ = 1$ right angle.

Again, $CS = PC - PS = PB - PA = AB = QR$

and $SR = CQ$ (by construction);

$\therefore$ $CQRS$ is a parallelogram of which $\angle QCS = 1$ right angle.

$\therefore$ CQRS is a rectangle.

$\therefore$ $SR \perp PS$ and $SR \perp QR$

$\therefore$ SR is a tangent to both the circles, that is, it is a transverse common tangent to two circles.

[*Remarks:* (1) Analysis of method of construction:

First method: The analysis of this method of construction is similar to that of the first method of construction of direct common tangent. So it is not restated.

Second Method:

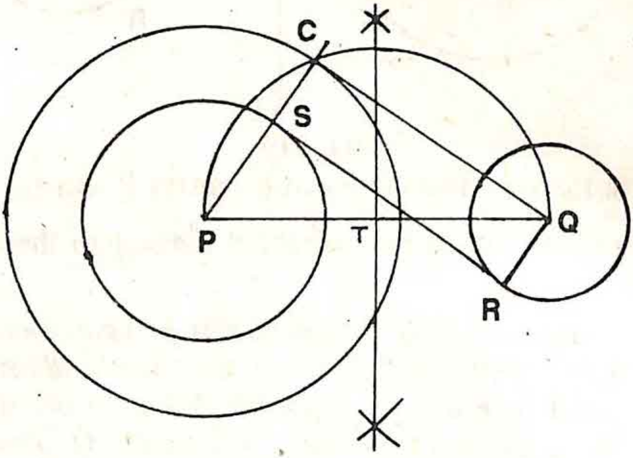


Fig. 220

Let SR be a transverse common tangent. Therefore, $PS \perp SR$, $QR \perp SR$; so $PS \parallel QR$. $QC \parallel RS$ is drawn:

Therefore, $CQRS$ is a rectangle. Now if the position of the point C is determined, the positions of the points S, R, that is, the common tangent SR can be obtained. From the rectangle $CQRS$, we get $QR = CS$. Therefore, the point C is the point of intersection of a new circle with centre P and radius equal to the sum of the radii of two circles and above mentioned semi-circle, for this that new circle and the semi-circle are to be drawn to determine the position of the point C. To avoid untidiness, instead of drawing the new circle and the semi-circle completely, two arcs are drawn in fig. 219. If only construction is required, C, Q need not be joined.

(2) If the two circles do not touch externally, another transverse common tangent can be drawn; that is, in that case, two transverse common tangents can be drawn to two circles.

(3) No transverse common tangent can be drawn to two intersecting circles.

(4) If the two circles touch each other externally or internally one common tangent can be drawn at the point of contact. We have already assumed this. We may call that common tangent as transverse common tangent in the first case and direct common tangent in the second case. The perpendicular drawn to the straight line, that joins the centres of two circles, at the point of contact is that common tangent.]

Some applications:

1. Take a line segment AB of length 8 cm. Draw two circles with centres A and B and radii 3 cm and 2 cm respectively. Draw two transverse common tangents to two circles.

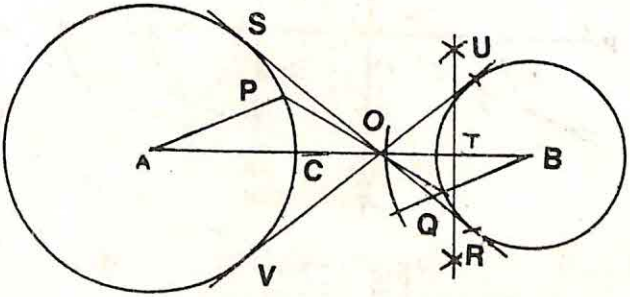


Fig. 221

Construction: A line segment AB of length 8 cm is taken. Circles are drawn, one with centre A and radius 3 cm and the other with centre B and radius 2 cm. AP, a radius of the circle with centre A, is drawn. BQ, a radius of the circle with centre B, is drawn making parallel to AP, so that P and Q are on the opposite sides of AB. P, Q are joined to meet AB at the point O. T, the mid-point of OB

is determined. An arc of a circle is drawn with centre *T* and radius *TB* to meet the circle with centre *B* at the point *R*. *R*, *O* are joined and *RO* is produced to touch the circle with centre *A* at the point *S*.

Then *RS* is a transverse common tangent to two circles.

Again, an arc of a circle is drawn with centre *O* and radius *OR* to meet the circle with centre *B* at the point *U*. *U*, *O* are joined and *UO* is produced to touch the circle with centre *A* at the point *V*.

Therefore, *UV* is the second transverse common tangent.

(Justify the method of construction.)

2. Draw two circles each of radius 2·5 cm. Draw two transverse common tangents to those two circles.

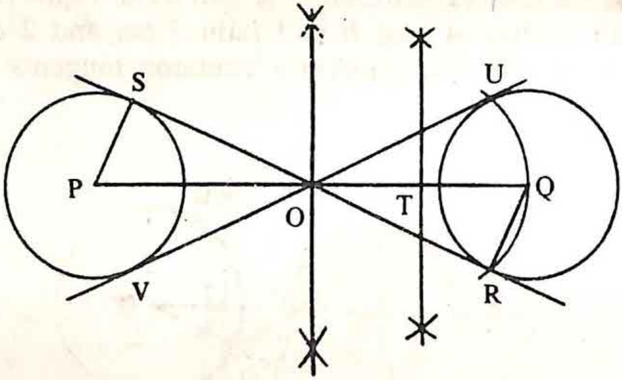


Fig. 122

Construction : Two circles are drawn with *P* and *Q* as centres and 2·5 cm radii. *P*, *Q* are joined. *O*, the mid point of *PQ*, and *T*, the mid point of *OQ* are determined. An arc of a circle is drawn with centre *T* and radius *TQ* to meet the circle with centre *Q* at the points *R* and *U*. *R*, *O* and *U*, *O* are joined, and, *RO* and *UO* are produced to touch the circle with centre *P* at the points *S* and *V* respectively.

Then *RS* and *UV* are the two transverse common tangents to the two circles.

Proof : $\because$ T is the mid point of OQ

$\therefore$ $\angle ORQ = 1$ right angle.

$\therefore$ $QR \perp OR$.

$\therefore$ the straight line RO is a tangent to the circle with centre Q.

Now, PS is drawn perpendicular to the straight line RO from the point P.

In $\triangle POS$ and $\triangle QOR$,

$\angle POS = \angle QOR$ (vertically opposite angles),

$\angle PSO = \angle QRO$ (each is 1 right angle),

$OP = OQ$ ($\because$ O is the mid-point of PQ)

$\therefore$ $POS \cong \triangle QOR$.

$\therefore$ $PS = QR =$ radius of the circle with centre P.

$\because$ radii of two circles are equal,

$\therefore$ RO, that is, the straight line ROS is also a tangent to the circle with centre P.

$\therefore$ the straight line ROS is a transverse common tangent to two circles.

Similarly, the straight line UOV is the other transverse common tangent to two circles.

Exercise 3.3

1. Draw two circles of radii 4 cm and 2·5 cm, their centres being 7·5 cm apart. Draw two transverse common tangents to those two circles.

2. Draw two circles, each of 3 cm radius and their centres being 10 cm apart. Draw a direct and a transverse common tangent to those two circles and find the lengths of those two tangents with the help of a scale.

3. Draw two circles of radii 4·4 cm and 1·6 cm, their centres being 6 cm apart. Draw all the common tangents to those two circles.

6·4 *Construction of mean-proportional ;*

CONSTRUCTION II

To draw the mean-proportional between two line segments.

First Method :

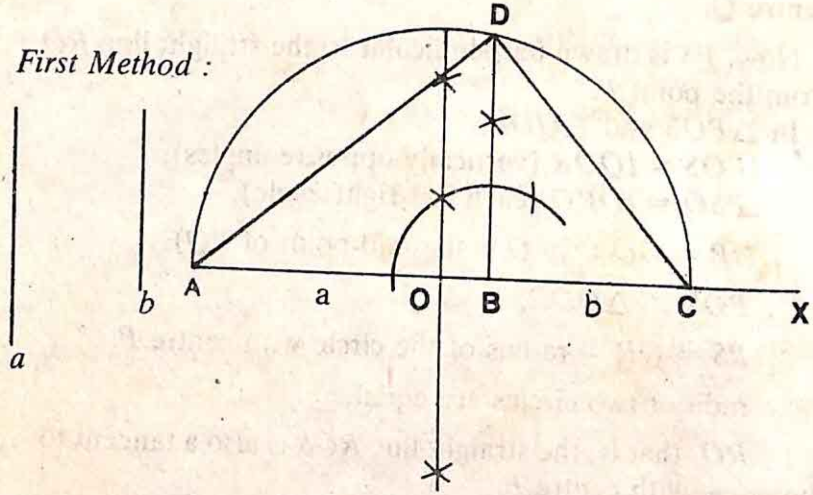


Fig. 223

Let *a*, *b* be two line-segments.

To draw the mean-proportional between these two line segments.

Construction: Any straight line *AX* is taken.

From *AX* the portion *AB* is cut off making equal to the line segment *a* and from *BX* the portion *BC* is cut off making equal to the line segment *b*.

A semi-circle is drawn with *AC* as diameter. The perpendicular is drawn to *BC* at the point *B* to meet the semi-circle at the point *D*.

Then *BD* is the required mean-proportional.

Proof: A, D and C, D are joined.

$\because \angle ADC$ is an angle in the semi-circle
$\therefore \angle ADC = 1$ right angle.

$\therefore DB$ is perpendicular to BC from the right angular point D of the right-angled triangle ADC.

$\therefore \triangle ABD$ and $\angle DBC$ are equiangular.

$$\frac{AB}{BD} = \frac{BD}{BC}$$

$\therefore BD^2 = AB.AC = a \,.\, b$
$\therefore BD$ is mean-proportional between a and b.

Second method :

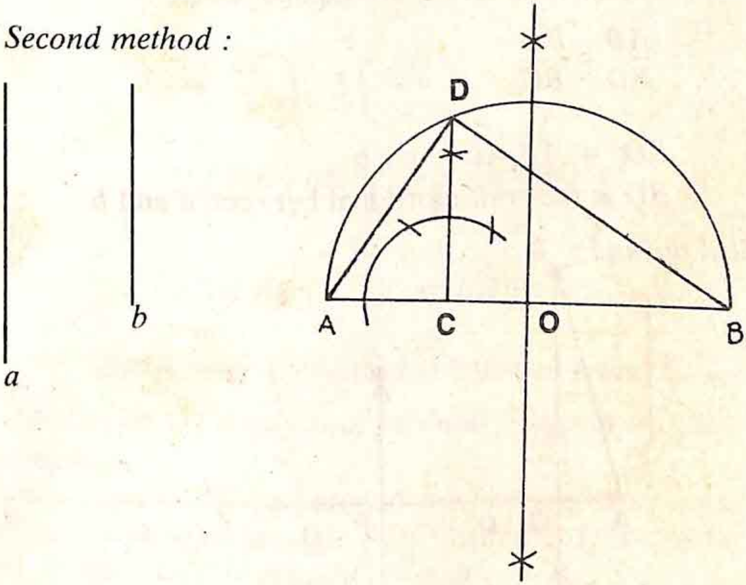


Fig. 224

Let a, b be two line-segments.

To draw the mean-proportional between these two line segments.

Construction: A line segment AB, equal to the line segment a, is taken. From BA a portion BC is cut off making equal to the line segment b. A semi-circle is drawn on AB. A perpendicular is drawn to AB at the point C to meet the semi-circle at the point D.

Then BD is the required mean-proportional.

Proof: A, D and B, D are joined.

$\because \angle ADB$ is an angle in the semi-circle
$\therefore \angle ADB = 1$ right angle.

$\therefore DC$ is perpendicular to AB from the right angular point D of the right-angled triangle ADB.

$\therefore \triangle ABD$ and $\angle DBC$ are equiangular.

$$\therefore \quad \frac{AB}{BD} = \frac{BD}{BC}$$

$\therefore BD^2 = AB.AC = a \,.\, b$

$\therefore BD$ is mean-proportional between a and b.

Third method :

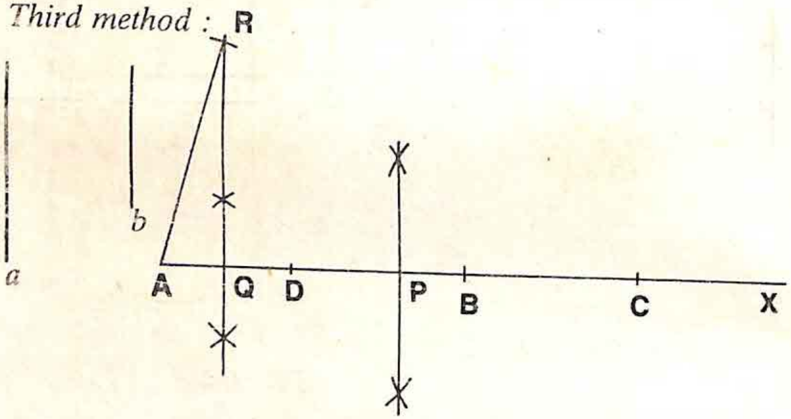


Fig. 225

Let a, b be two line-segments.

To draw the mean-proportional between these two line segments.

Construction: Any straight line AX is taken. From

AX a portion AB is cut off making equal to the line segment a. From that straight line the portions, BC and BD are cut off on both sides of B, making equal to the line segment b. Perpendicular bisectors of AC and AD are drawn to meet the straight line AX at the points P and Q respectively. An arc of a circle with centre A and radius AP is drawn to meet the perpendicular bisector of AD at the point R.

Then QR is the required mean-proportional.

Proof: A, R are joined.

$\because$ QR is the perpendicular bisector of AD.

$\therefore$ $\triangle AQR$ is a right-angled triangle.

$$\therefore QR^2 = AR^2 - AQ^2$$
$$= AP^2 - AQ^2$$
$$= \left(\frac{AC}{2}\right)^2 - \left(\frac{AD}{2}\right)^2$$
$$= \left(\frac{AB+BC}{2}\right)^2 - \left(\frac{AB-BD}{2}\right)^2$$
$$= AB.BC \; [\because BC = BD]$$
$$= a.b$$

$\therefore$ QR is mean-proportional between a and b.

[*Remarks:* (1) *Analysis of methods of constructions*

First method:

We know if DB be perpendicular to the hypotenuse AC of a right-angled triangle ADC from the right angular point D, then $DB^2 = AB.\ BC = a.\ b$

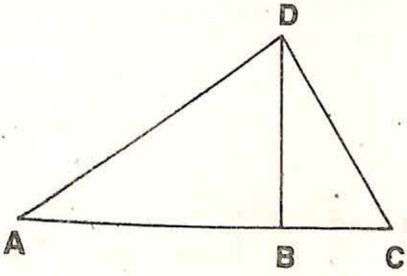


Fig. 226

Again, $AC = AB + BC = a+b$,

$\therefore$ the point B is obtained, if we take the line segment AC as the sum of the line segments a and b. Now the mean-proportional is obtained, if the point D is obtained. Since $\angle ADC = 1$ right angle, therefore, the point D is on the semi-circle drawn on AC. Again, the point D is also on the perpendicular drawn to AC at the point B. Therefore, the point D is the point of intersection of the semi-circle drawn on AC and the perpendicular drawn to that straight line at the point B. This suggests the method of construction.

Second method:

We know if DC is perpendicular to AB from the right angular point D of the right-angled triangle ADB, then $BD^2 = AB.\ BC = a \cdot b$ Again $AC = AB - BC = a - b$

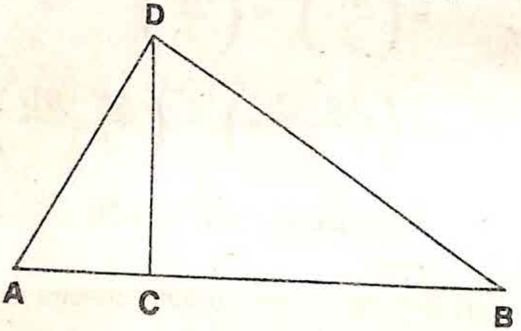


Fig. 227

$\therefore$ AC is the difference of a and b. Thus the point c is obtained. Now the mean-proportional is obtained, if the point D is obtained. The analysis of the first method of construction suggests the procedure for obtaining the point D.

Third method:

$$ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$$

∴ if x be the mean-proportional between a and b; then

$$x^2 = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$$

$$Or,\ x^2 + \left(\frac{a-b}{2}\right)^2 = \left(\frac{a+b}{2}\right)^2$$

∴ $x, \frac{a-b}{2}, \frac{a+b}{2}$ are the sides of a right-angled triangle.

The hypotensuse $\frac{a+b}{2}$ and another side $\frac{a-b}{2}$ of the right-angled triangle are known.

∴ x, that is, the mean-proportional is obtained if the right-angled triangle is drawn.

(2) We know how to find the mean-proportional between the line segments a and b. This means we are now in a position to find the side of a square, the area of which is equal to the rectangle contained by the adjacent sides a and b. Therefore, the above methods enable us to draw a square equal in area to a rectangle.

(3) If a and b be the magnitudes of two line segments and if x be mean-proportional between them, then $x^2 = ab$, that is $x = \sqrt{ab}$. Therefore, it is observed that if we resolve a number into two factors, then the mean-proportional between those two factors is the square root of the number. Therefore, we can find the square root of a number geometrically with the help of drawing the mean-proportional.]

Sufficient ideas about these things will be obtained through the following applications.]

Some applications.

1. Draw a square equal in area to a rectangle of length 9 cm and breadth 4 cm

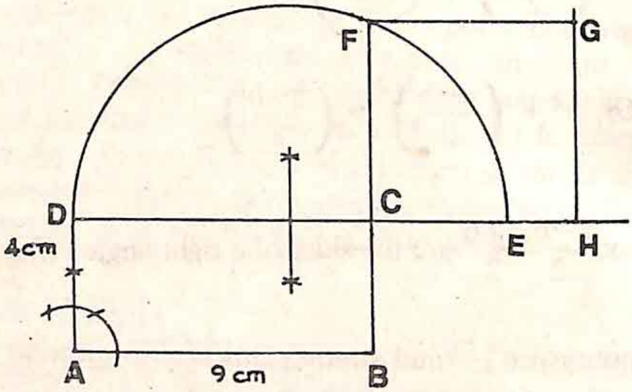


Fig. 228

Construction: The rectangle $ABCD$ of length 9 cm and breadth 4 cm is drawn. DC is produced and from DC produced the portion CE is cut off making equal to CB. A semi-circle is drawn on DE. BC is produced to meet the semi-circle at the point F. The square $CFGH$ drawn with CF as one of its sides.

Then, $CFGH$ is the required square.

Proof: According to construction $ABCD$ is the given rectangle.

Again, $\because \angle BCD = 1$ right angle,
$\therefore CF$ is perpendicular to DE

$\therefore$ according to construction, CF is mean- proportional between DC and CE.

$$\therefore CF^2 = DC.CE$$
$$= AB.BC$$
$$= \text{Area of the rectangle.}$$

2. Find, geometrically, the value of $\sqrt{21}$
Or, Find the square root of 21, geomertrically.

First method:

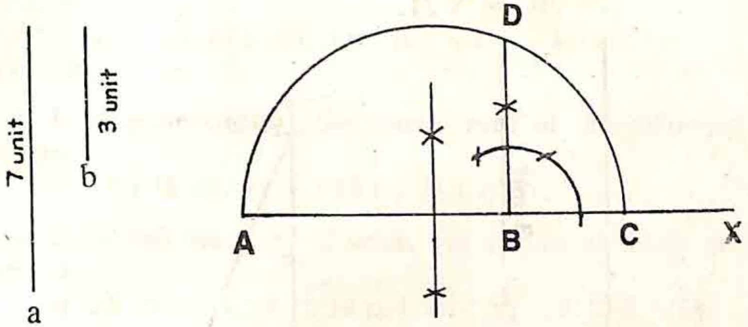


Fig. 229

Construction: Two line segments a and b of lengths 7 units and 3 units respectively are taken. From a straight line AX the portions AB and BC are cut off making equal to the line segments a and b respectively on the same side.

BD, the mean-proportional of AB and BC is drawn.

Then the length of the line segment BD denotes the value $\surd 21$ or the square root of 21. It is found, with the help of a scale that $BD = 4{\cdot}6$ units (approximately).

Proof: $\because$ BD is the mean-proportional between AB and BC.

$$\begin{aligned}\therefore\ BD^2 &= AB.BC\\ &= a.b\\ &= 7.3\\ &= 21.\\ \therefore\ BD &= \surd 21.\end{aligned}$$

Second method:

Construction: A right-angled triangle ABC is drawn with sides $AB = 2$ units and hypotenuse $BC = 5$ units.

Then the length of AC is $\surd 21$ units.

It is found, with the help of a scale, that $AC = 4{\cdot}6$ units (approximately).

Proof: $\because$ $\triangle ABC$ is a right-angled triangle,

$$\begin{aligned}\therefore\ AC^2 &= BC^2 - AB^2\\ &= 5^2 - 2^2\\ &= 25 - 4\end{aligned}$$

$$= 21.$$

$$\therefore AC = \sqrt{21}.$$

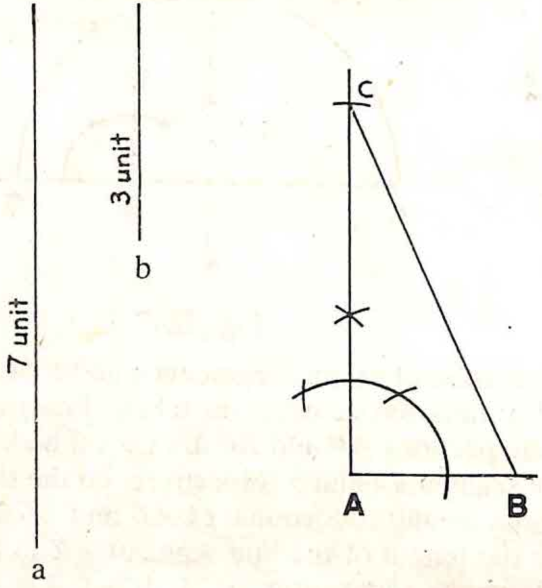


Fig. 230

[*Remark: A* square with side $\sqrt{21}$ could have been drawn also following the same method. In that case a square would have to be drawn with AC as a side.]

Exercise 34

[State the method of construction and give proof in each case.]

1. Draw the mean-proportional between the line segments of following lengths:

(i) 4·5 cm, 2 cm *(ii)*. 4 cm, 3 cm *(iii)* 8 cm, 4 cm *(iv)* 16 cm, 2 cm. *(v)* 10 cm, 3 cm *(vi)* 12 cm, 3 cm.

Find the value of the mean-proportional with the help of a scale, in each case.

2. Draw the square (s) equal in area to the rectangle (s) of sides given below:

(i) 8 cm, 6 cm *(ii)* 5 cm, 3 cm *(iii)* 4·2 cm, 3·5 cm.
(iv) 8·9 cm, 3·1 cm.

3. Find geometrically, the square root of the following numbers:

(i) 7 *(ii)* 15 *(iii)* 12 *(iv)* 18 *(v)* 24 *(vi)* 30

4. Draw the square(s) of sides, the lengths of which are given below:

(i) $\sqrt{8}$ *(ii)* $\sqrt{11}$ *(iii)* $\sqrt{14}$ *(iv)* $\sqrt{22}$ *(v)* $\sqrt{27}$ *(vi)* $\sqrt{28}$.

5. Draw the square(s) equal in area to the following triangle(s):

(i) The three sides of the triangle are of lengths 10 cm, 7 cm and 5 cm.

(ii) The triangle is isosceles having the length of the base 6 cm and the length of each equal side 5 cm.

(iii) The triangle is equilateral having length of each side 7 cm.

Miscellaneous Exercise 7

Calculations, if necessary, should be done. Give reasons in support of your answer in each case.

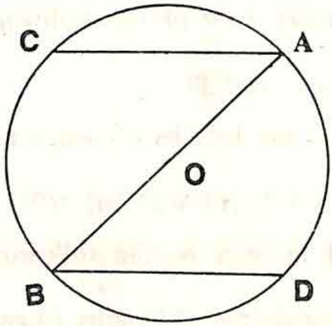


Fig. 231

1. In fig. 231 AB is the diameter. $AC \parallel BD$. $AC = 4$ cm. $BD = ?$

If O be the centre of the circle, does the line segment CD pass through the point O?

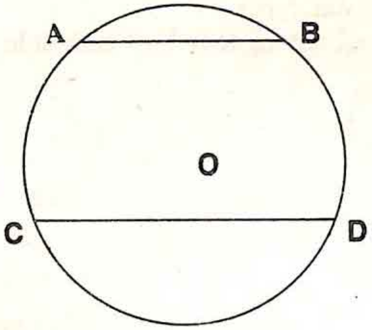


Fig. 232

2. In fig. 232 $AB \parallel CD$; $OA = 5$ cm, $AB = 6$ cm, $CD = 8$ cm. What is the distance between AB and CD?

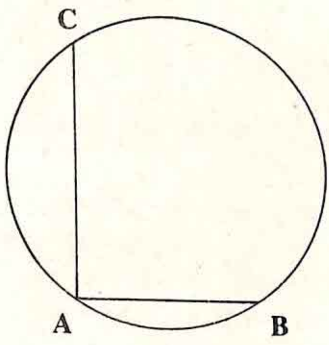


Fig. 233

3. In fig. 233 $AB \perp AC$ and $AB = 5$ cm, $AC = 12$ cm. What is the length of the radius of the circle?

4. In fig. 234, two circles are concentric. If $AB=2$ cm, CD = how much ?

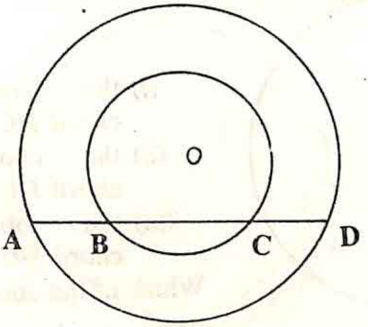


Fig. 234

5. In fig. 235, $AB \parallel CD$; $AB=10$ cm, $CD=24$ cm, the distance between AB and CD is 17 cm. What is the length of the radius of the circle?

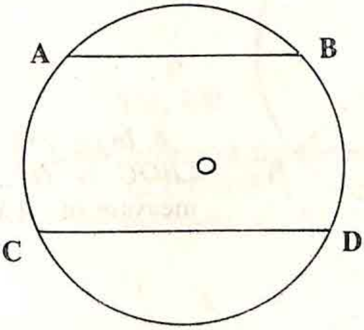


Fig. 235

6. In fig. 236, C is the mid-point of the arc ACB of the circle. $CD \perp AB$, if $CD=2$ cm, $AB=12$ cm, what is the length of the radius of the circle?

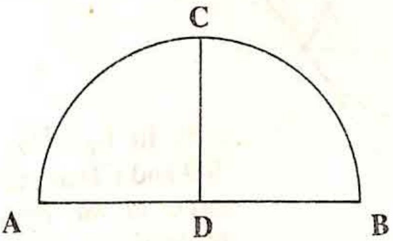


Fig. 236

7. In fig. 237, if AD is the bisector of $\angle BAC$,

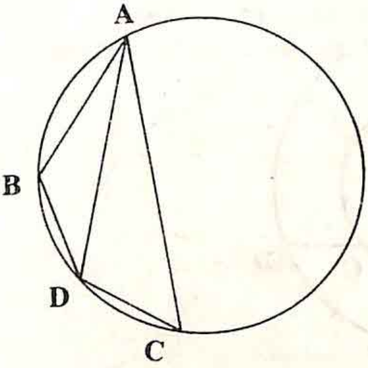


Fig. 237

(i) the chord $BD >$ the chord DC,
(ii) the chord $BD =$ the chord DC,
(iii) the chord $BD <$ the chord DC.

Which of the above relations is true?

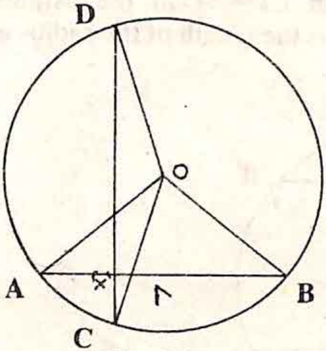


Fig. 238

8. In fig. 238, $\angle AOD = 110°$, $\angle BOC = 68°$. What is the measure of $\angle AXC$?

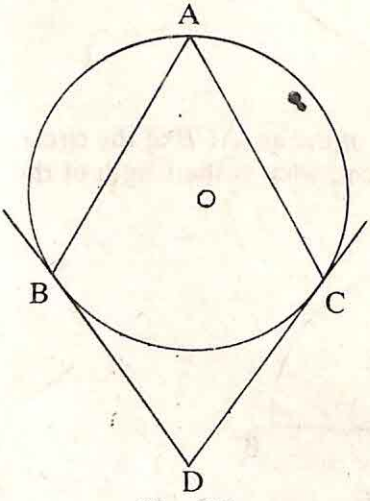


Fig. 239

9. In fig. 239, $\angle BAC = 55°$. BD and CD are tangents to the circle at the points B and C respectively. What is the measure of $\angle BDC$?

10. In fig. 240, PA and PB are two tangents to the circle. If $\angle APB = 48°$, what is the mea sure of $\angle AEB$?

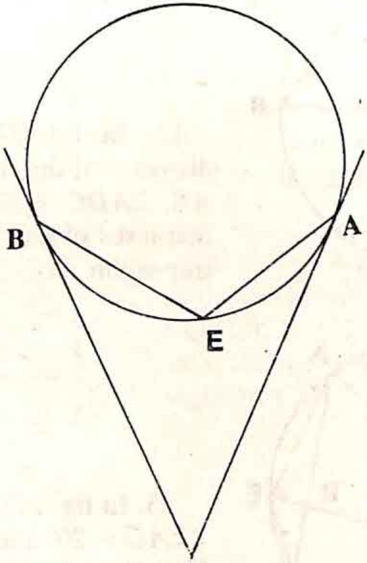


Fig. 240

11. In fig. 241, O is the centre of the circle and $ABCD$ is a cyclic quadrilateral. If $\angle COD = 130°$, $\angle BAC = 25°$ and $\angle BCA = 40°$, what are the measures of $\angle BCD$ and $\angle ADC$?

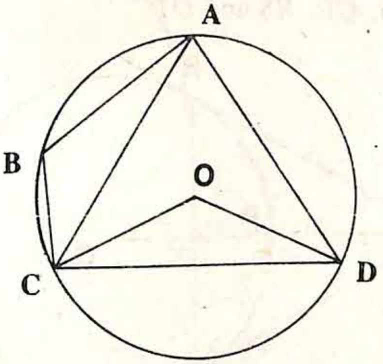


Fig. 241

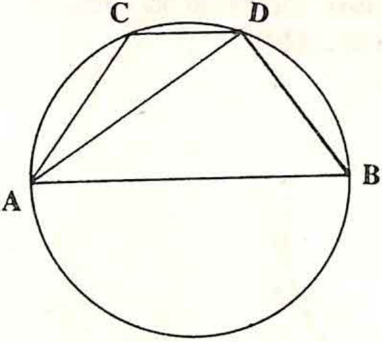


Fig. 242

12. In fig. 242, AB is a diameter of the circle and $CD \parallel AB$, $\angle ADC = 35°$. Find the measures of each angle of the trapezium $ABDC$.

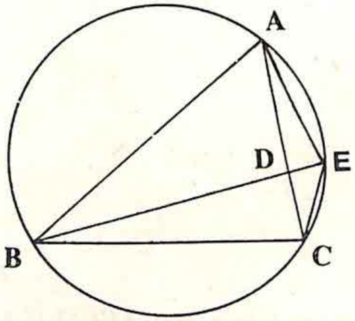


Fig. 243

13. In fig. 243, $BE \perp AC$. If $\angle EAC = 20°$ and $\angle BEC = 60°$, find the measures of each of the angles of $\triangle BAC$ and $\triangle AEC$.

14. In fig. 244, OP=6 cm, PQ=4 cm, QR and QS are two tangents, and O is the centre of the circle. Find the lengths of the line segments QS, QR, RS and OT.

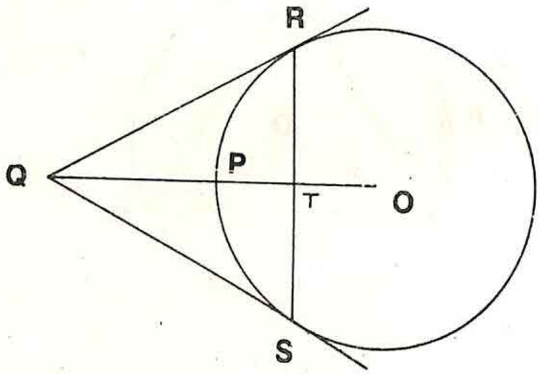


Fig. 244

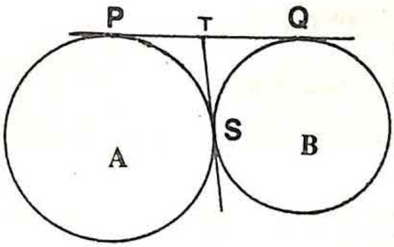


Fig. 245

15. In fig. 245, two circles with centres A and B touch each other externally at the point S. PQ and ST are common tangents. The length of the radius of the circle with centre A is 9 cm and that of the circle with centre B is 4 cm. Find the lengths of PQ and ST.

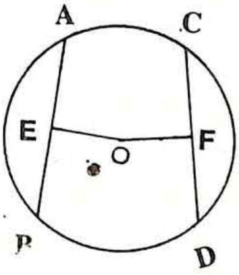


Fig. 246

16. In fig 246, O is the centre of the circle and $OE \perp AB$ OF $\perp$ CD, OE = OF. If AE=2·5 cm, what is the length of the line segment CD?

17. In fig. 247, three circles with centres A, B and C touch each other externally. The radii of three circles are 3 cm, 2·8 cm and 1·2 cm respectively. Find the lengths of the sides of $\triangle ABC$ and state what kind of triangle it is.

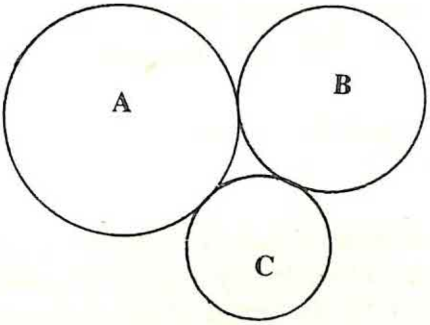


Fig. 247

Miscellaneous Exercise 8

1. Fill in the gaps with appropriate words :

(i) At least — non-collinear points are required to draw a definite circle.

(ii) If perpendicular is drawn to any chord of a circle from the centre of the circle, the perpendicular — the chord.

(iii) The measure of the angle formed by the line segment that joins the mid-point of any chord of a circle and the centre of the circle with the chord is —.

(iv) In a circle, the angle at the circumference is — the angle at the centre standing on the same arc.

(v) The angles in the same segment are — each other.

(vi) The angle in the semi-circle is —, the angle in the segment of a circle greater than a semi-circle is — and the angle in the segment of a circle lesser than a semi-circle is —.

(vii) Two opposite angles of a cyclic quadrilateral are —.

(viii) Two points C and D lie on the same side of the line segment AB and $\angle ACB$ is equal to $\angle ADB$. Four points A, B, C, D are —.

(ix) In a quadrilateral $ABCD$, $\angle ABC$ and $\angle ADC$ are together equal to 180°; The quadrilateral $ABCD$ is a — quadrilateral.

(x) The tangent to a circle is — to the radius through the point of contact.

(xi) The tangents drawn to a circle from a point outside it are —, the angles subtended by them at the centre are —, the straight line that joins the centre of the circle and the exterior point is the — of the angle included by the tangents and that straight line — the chord of contact —.

(xii) If two circles touch each other externally, their centres and the point of contact lie on the same — — and the distance between the two centres is — to the — of the two —

(xiii) If two circles touch each other internally, their centres and the point of contact are — and the distance between the two centres is — to the — of the two —.

(xiv) The straight line parallel to a side of a triangle divides other two sides —.

(xv) If two sides of a triangle be divided proportionally, the straight line that joins the points of division is — to the third side.

(xvi) The — sides of two equiangular triangles are —.

(xvii) If the sides of two triangles be —, those two triangles are similar.

(xviii) Each of the two triangles formed by dropping a perpendicular on the hypotenuse of a right-angled triangle from the right angular point is — to the original triangle and they — are —.

(xix) The square on the — of a right-angled triangle is — to the — of the — on other two sides.

(xx) If the square on the greatest side of a triangle be equal to the sum of the squares on other two sides, the angle opposite to the greatest side is — angle.

2. Fill in the gaps with appropriate words :

(i) The bisector of an angle at the circumference standing on an arc of a circle — that arc.

(ii) The bisectors of all angles standing on an arc of a circle are — and that is — — the circle.

(iii) A cyclic trapezium is an — trapezium.

(iv) A cyclic parallelogram is a —

(v) The diagonal of a cyclic square is a — of the circle.

(vi) The two tangents drawn at the end points of a diameter of a circle are —.

(vii) The straight line that joins the centres of two intersecting circles is the — — of the common chord.

(viii) The two direct common tangents of two circles are — in length.

(ix) If two non-parallel line segments be the chords of a circle, the — of — of the — — of those two line segments is the centre of that circle.

(x) If two intersecting straight lines be the tangents to a circle, the — of the angle included by those two straight lines passes through the centre of the circle.

3. Write down the question numbers of the correct statements in your exercise book and give a '√' mark against them, and write down the incorrect statements correctly :

(i) Only one circle can be drawn through any three points.

(ii) Any chord perpendicular to a diameter of a circle is bisected by the diameter.

(iii) An angle at the centre is double the angle at the circumference.

(iv) The sides of equiangular triangles are proportional.

(v) The perpendicular drawn to the end point of a radius of a circle is a tangent to the circle at that point.

(vi) Tangents to a circle at any two points on the circle must intersect each other.

(vii) If three congruent circles touch each other externally their centres are the vertices of an equilateral triangle.

(viii) If two tangents to a circle from a point outside the circle are at right angles, a square is formed by joining in order the centre, a point of contact, the exterior point and the other point of contact.

(ix) If two triangles are equiangular, then these two triangles are similar.

(x) The circle drawn with hypotenuse of a right-angled triangle as diameter must pass through right angular point.

Miscellaneous Exercise 9

1. *ABC* is a triangle. How are the sides of that triangle related to the circum-circle of the triangle? From that relation, state how the centre of the circle can be determined and what the radius of that circle is. Now, draw a triangle *ABC* and draw the circum-circle of that triangle.

2. *ABC* is a triangle. How are the sides of that triangle related to the in-circle of that triangle? State, from these how the centre of the circle can be determined and what the radius of that circle is. Now draw a triangle *ABC* and draw the in-circle of that triangle.

3. *OP* is a radius of a circle with centre *O*. How is that radius related to the tangent to the circle at the point *P* ? Using this, draw a tangent to a circle with centre *O* at a point *P*.

4. *O* is the centre of a circle and *P* is a point outside the circle. If a tangent to the circle is drawn from the point *P*, what is the measure of the angle that *OP* makes at the point of contact? What is the measure of an angle that a line segment subtends at any point on a semi-circle drawn with that line segment as diameter? Using these, draw two tangents to a circle with centre *O* from a point *P* outside the circle.

5. *A* and *B* are the centres of two circles. *PQ* is a direct common tangent. $BR \parallel QP$.

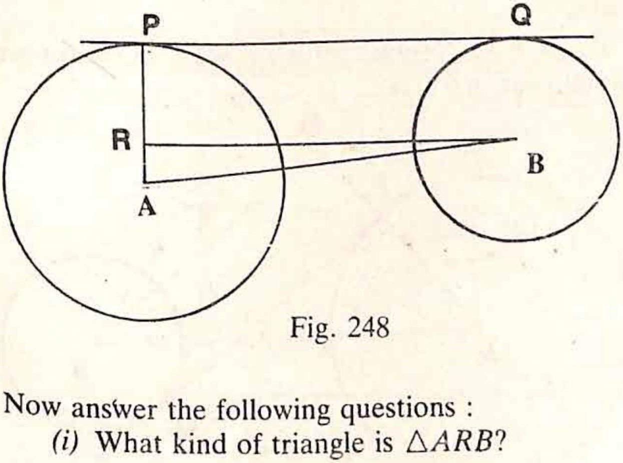


Fig. 248

Now answer the following questions :

(*i*) What kind of triangle is $\triangle ARB$?

(ii) What is the length of the line segment AR?

(iii) How is the point R obtained?

(iv) How is the point P determined, if the point R is obtained?

(v) How can the common tangent be obtained, if the point P is obtained?

Using these draw a direct direct common tangent to two circles with centres A and B.

6. A and B are the centres of two circles. AC and BD are two parallel radii. Both CD and AB produced meet each other at the point O. The tangent to circle with centre B, is drawn from the point O. Show, with reasons, that OQ produced will also touch the circle with centre A.

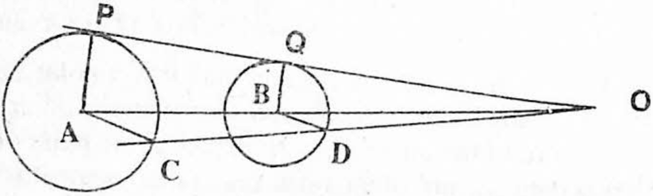


Fig. 249

Draw a direct common tangent applying this method.

7. Are these two methods applicable to two circles of equal radii? What kind of quadrilateral will $ABQP$ be in that case? From this datum draw a direct common tangent to two circles of equal radii.

8. A and B are centres of two circles. PQ is a transverse common tangent. $BR \parallel QP$.

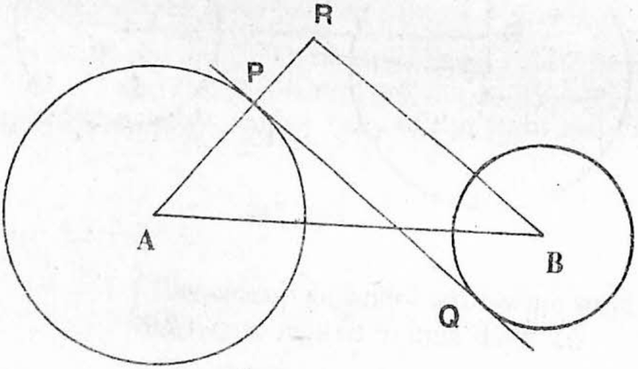


Fig. 250

Now answer the following questions :

(i) What kind of triangle is $\triangle ARB$?

(ii) What is the length of the line segment AR?

(iii) How can the point R be determined?

(iv) How can the point P be determined, if the point R is obtained?

(v) How can the common tangent PQ be obtained, if the point P is obtained?

Using these data, draw transverse common tangent to two circles with centres A and B.

9. A and B are the centres of two circles. AC and BD are two parallel radii. CD and AB meet at the point O. OQ, a tangent to the circle with centre B, is drawn from the point O. Show, with reasons, QO produced will also touch the circle with centre A.

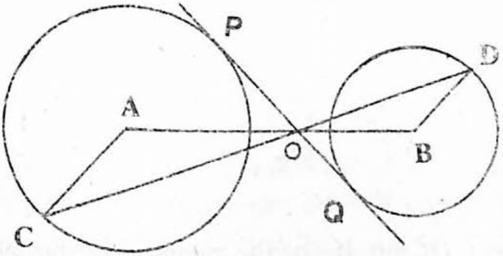


Fig. 251

Draw a transverse common tangent applying this method.

10. What do you understand by 'mean-proportional'? If c be the mean-proportional between a and b, what is the relation between a, b, c?

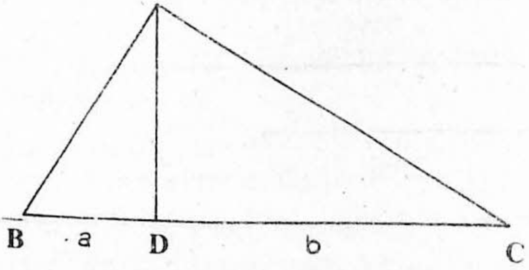


Fig. 252

ABC is a right-angled triangle with BC as hyoptenuse. $AD \perp$ BC. Which line segment is the mean-proportional between BD and DC?

Let $BD=a$, $DC=b$. If c be the mean-proportional between a and b, which line segment denotes c ? How are a and b related with BC ? What is to be drawn to construct a right-angled triangle with BC as hypotenuse? How is the point A obtained after the above construction? Using the above data, draw the mean-proportional between a and b.

11. ABC is a right-angled triangle with BC as hypotenuse. $AD \perp$ BC.

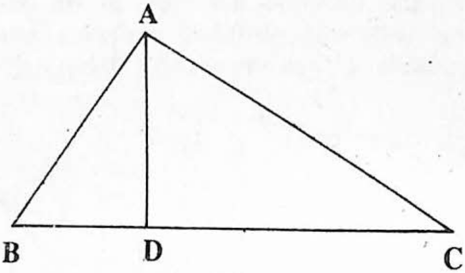


Fig. 253

Let $BC=a$, $DC=b$. If c be the mean-proportional between a and b; which line segment does denote c? How are a and b related with c ?

What do you need to construct a right-angled triangle with BC as hypotenuse? How can the point A be obtained after this construction.

Using these data, draw the mean-proportional between two line segments a and b $(a>b)$

12. a ________________________

b ________________

$a + b$ ____________________________________

$a - b$ __________

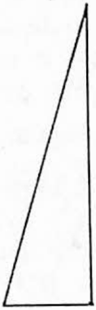
a and b are two line segments.

From the relation

$$\left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 = ab,$$

state what kind of triangle can be drawn with sides

$$\frac{a+b}{2}, \frac{a-b}{2} \text{ and } \sqrt{ab}$$



How are a and b related with $\sqrt{ab}$?

Fig. 254

Using these data draw the mean-proportional between the line segments a and b.

13. Using the method of drawing mean-proportional find, geometrically, the square rooot of 24.

Miscellaneous Exercise 10

1. Two chords AB and CD of a circle meet each other at the point P. If $PB = PD$, prove that, $AB = CD$; also prove that, $ACBD$ is a trapezium.

2. AB and CD are two parallel chords of a circle. AD and BC meet at the point P. Prove that, $PC = PD$.

3. O is the in-centre of $\triangle ABC$. AO produced meets the circum-circle of that triangle at the point D. Prove that, $DB = DC = DO$.

4. Six points A, B, C, D, E, F lie on a circle. Prove that., $\angle ABC + \angle CDE + \angle EFA = 4$ right angles.

[Hints: Draw $\triangle ACE$ and use the properties of a cyclic quadrilateral].

5. The point of intersection of two diagonals of a cyclic parallelogram is the centre of the circle—prove.

6. The side BC of a cyclic quadrilateral $ABCD$ is produced to the point E. Prove that the bisectors of $\angle BAD$ and $\angle DCE$ meet at a point on the circle.

7. The bisectors of $\angle A$, $\angle B$ and $\angle C$ of $\triangle ABC$ meet the circum-circle of the triangle at the points X, Y and Z respectively. Prove that, $YZ \perp AX$.

[Hints: Join A, Y and prove that the sum of $\angle XAY$ and $\angle AYZ$ is equal to $\frac{1}{2}(\angle A + \angle B + \angle C)$].

8. A tangent PA and a secant PCD are drawn to a circle with centre O from a point P outside the circle. Join A, P, B with the point O. Draw OD perpendicular to BC and prove that, $AP^2 = PD^2 - BD^2$. Hence prove that, $AP^2 = PB.\ PC$.

9. PCD is a secant drawn to a circle with centre O from a point P outside the circle. AB is a diameter. The perpendicular to AB drawn from the point P meets AB produced at the point M. Prove that, $PM^2 = PC.\ PD - MB.\ MA$.

10. P, Q, R, S, T are such five points on a circle that $\angle QPR = \angle RPS = \angle SPT$. Prove that, the angle contained by two chords QS and PR is equal to that contained by QT and PS.

11. Two chords AB and CD of a circle with centre O meet each other at the point P. The tangents to the circle at the points A and B meet at the point Q and those at the points C and D meet at the point R. Prove that, $\angle AQB + \angle CRD = 2\angle BPC$.

12. In a cyclic quadrilateral $ABCD$, the sides AB and DC, when produced, meet at the point Q and the sides BC and AD, when produced, meet at the point R. The bisectors of $\angle BQC$ and $\angle CRD$ meet each other at the point O. Prove that,

(i) $\angle BQC + \angle CRD = 180° - 2\angle BAD$,

(ii) $\angle QOR = \angle BCD - \frac{1}{2}(\angle BQC + \angle CRD)$

Hence prove that, $\angle QOR = 1$ right angle.

13. $\triangle ABC$ is inscribed in a circle. The perpendicular drawn to AC from the point B meets AC at the point D and the circle at the point F. The perpendicular drawn to AB from the point C meets AB at the point E and the circle at the point G. Prove that, $FG \parallel DE$.

14. $\triangle ABC$ is inscribed in a circle. The bisector of $\angle BAC$ meets the circle at the point D. The circle with centre D and radius DC meets AD at the point P. Prove that, BP is the bisector of $\angle ABC$.

[Hints: Prove that CP is the bisector of $\angle ACB$].

15. A direct common tangent and a transverse common tangent to two circles with centres A and B meet each other at the point P. Prove that the point P is on the circle with AB as diameter.

16. P and Q are two points on a straight line. PR and QS are perpendiculars to the straight line at the points P and Q respectively. PS and QR meet each other at the point O. OT is perpendicular to PQ. Prove that,

$$\frac{1}{OT} = \frac{1}{PR} + \frac{1}{QS}.$$

17. $ABCD$ is a rhombus. A straight line through the point D meets AB and DA produced at the points P and Q respectively. Prove that, $\triangle APQ$, $\triangle PBC$ and $\triangle DCQ$ are equiangular. Hence prove that, $PB : DQ = AP^2 : AQ^2$.

18. In a cyclic quadrilateral $ABCD$, the sides BA and CD, when produced, meet at the point P. The bisector of $\angle BPC$ meets AD and BC at the points Q and R respectively. Prove that $AQ : QD = RC : BR$.

19. A straight line parallel to the side BC of $\triangle ABC$ intersects AB and AC at the points P and Q respectively. BQ and CP meet each other at the point R. AR produced meets BC at the point S. Prove that, S is the mid-point of BC.

[Hints: Draw parallel to PC through the point B and produce AS to meet this parallel straight line. Join C with the point of intersection. Now proceed.]

20. D and E are any two points on the sides AB and BC of $\triangle ABC$. A straight line parallel to BC through the point D meets AC at the point H. F and G are two such points on AB such that $\frac{BE}{DH} = \frac{BG}{DF}$. Prove that, $FH \parallel EG$.

21. D is any point on the side BC of $\triangle ABC$. Straight lines drawn through the point D parallel to AB and AC meet AC and AB at the points E and F respectively. Prove that—

$$\frac{\triangle FBD}{\triangle AEF} = \frac{\triangle AEF}{\triangle EDC}.$$

22. $\triangle ABC$ is inscribed in a circle. AD is a diameter and AE is perpendicular to BC. Prove that, $\triangle AEB$ and $\triangle ACD$ are

equiangular. Hence prove that, $AB. AC = AE. AD$.
(Brahmagupta's Theorem).

23. ABC is a triangle. Three concurrent straight lines AD, BE and CF meet at the point O. D, E, F lie on BC, CA and AB respectively. Prove that,

$$\frac{BD}{DC} = \frac{\triangle ABD}{\triangle ADC} = \frac{\triangle OBD}{\triangle ODC} = \frac{\triangle AOB}{\triangle AOC}$$

Find similar two relations for CE, EA and AF, FB and prove that, $\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = 1$

(Ceva's Theorem—1678 A.D.).

24. A straight line meets the sides BC, CA and BA produced of $\triangle ABC$ at the points L, M and N respectively. AP, BQ and CR are perpendiculars to the straight line LMN from the points A, B and C respectively. Prove that, $\triangle BLQ$ is similar to $\triangle CLR$, $\triangle CRM$ to $\triangle APM$ and $\triangle APN$ to $\triangle BLN$. Hence prove that,

$$\frac{BL}{CL}\cdot\frac{CM}{AM}\cdot\frac{AN}{BN} = 1$$

(Menelaus' Theorem—About 100 A.D.)

25. From a point P on the circum-circle of $\triangle ABC$, PL, PM and PN are perpendiculars to AB, BC and CA produced respectively; prove that, *(i)* $\angle PLM = 180° - \angle PBC$, *(ii)* $\angle PLN = \angle PAN$, *(iii)* $\angle PLM + \angle PLN = 180°$, *(iv)* MLN is a straight line.
[The straight line MLN is called Simson's line].

26. O is the ortho-centre of $\triangle ABC$. Prove that, the point O is the in-centre of the pedal triangle.

27. If a straight line parallel to the side BC of $\triangle ABC$ meets AB and AC at the points P and Q respectively, then $\frac{AP}{PB} = \frac{AQ}{QC}$.

A Student proved this theorem in the following-way:

$\because$ $PQ \parallel BC$ and APB is a transversal
$\therefore$ $\angle APQ =$ corresponding $\angle ABC$,
Similarly, $\angle AQP = \angle ACB$,
$\angle PAQ = \angle BAC$ (same angle).

$\therefore$ $\triangle APQ$ and $\triangle ABC$ are equiangular.

$\therefore$ Corresponding sides are proportional,

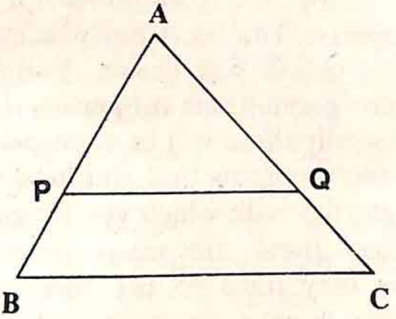


Fig. 255

That is, $\dfrac{AP}{AB} = \dfrac{AQ}{AC}$,

$$\therefore \frac{AP}{PB} = \frac{AQ}{QC}$$

Is the proof correct? If not, state where is the defect?

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APPENDIX

You have gone through a table in class VIII. From the table you know the conditions for obtaining a geometrical property. That is, from which data you can arrive at the conclusion you desire. You have become familiar with some geometrical properties in classes IX & X, for which naturally there will be some additions in that table. Besides, the theorems that you have learnt, or the geometrical properties with which you have been familiar through exercises, there are many other geometrical properties. You may have to use those properties in future, particularly in case, when you will get with higher education. Another table has been given here from which you will be able to know the properties of different figures. The following two tables will help you much in knowing or establishing those geometrical properties:

First Table

Given figure or figures	The properties we get
1. If a line segment stands on a straight line, the adjacent angles so formed are equal.	(1) Each of the angles is a right angle (from definition). (2) One of the straight lines is perpendicular to the other (from definition).
2. A line segment stands on a straight line.	The sum of the two adjacent angles is equal to two right angles.
3. The sum of two adjacent angles is equal to two right angles.	Exterior arms of the two adjacent angles are on the same straight line.

Given figure or figures	The properties we get
4. Two straight lines intersect each other.	The vertically opposite angles are equal to each other.
5. One of the two straight lines is the internal bisector of an angle and the other is the external bisector of that angle.	(1) The two straight lines are perpendicular to each other. (2) The angle included by the two bisectors is a right-angle.
6. Two straight lines are perpendicular to each other.	Each of the angles is a right angle (from definition).
7. An angle is a right angle.	The arms of the angle are perpendicular to each other (from definition).
8. Two straight lines are parallel.	Two straight lines have no point of intersection (from definition).
9. A straight line intersects two parallel straight lines.	(1) The corresponding angles are equal to each other. (2) The alternate angles are equal to each other. (3) Two interior angles on one side of the transversal are together equal to two right angles.
10. A straight line intersects three parallel straight lines so that two intercepts of the transversal are equal.	Two intercepts of any other transversal are also equal.

Given figure or figures	The properties we get
11. A straight line intersects two other straight lines so that *(i)* two corresponding angles are equal, or, *(ii)* two alternate angles are equal, or, *(iii)* the two interior angles on one side of the transversal are together equal to two right angles.	Those two straight lines are parallel to each other.
12. A straight line is parallel to any of two intersecting straight lines.	That straight line cannot be parallel to other intersecting line.
13. Of all the line segments drawn from a point outside a straight line to that straight line, one is the least.	That line segment is perpendicular to the straight line.
14. A triangle.	(1) The sum of the lengths of any two sides is greater than the length of the third side. (2) The sum of three angles is equal to two right angles.
15. One side of a triangle is produced.	(1) The exterior angle is equal to the sum of two interior opposite angles. (2) The exterior angle is greater than any of the interior opposite angles.

Given figure or figures	The properties we get
16. One side of a triangle is greater than another.	The angle opposite to the greater side is lesser than the angle opposite to the less.
17. One angle of a triangle is greater than another.	The side opposite to the greater angle is lesser than the side opposite to the less.
18. An isosceles triangle.	(1) Two sides are equal (from definition). (2) The angles opposite to the equal sides are equal. (3) The bisector of the vertical angle is the per pendicular bisector of the base.
19. Two angles of a triangle are equal.	The sides opposite to the equal angles are equal.
20. An equilateral triangle.	(1) Three sides are equal (from definition). (2) Three angles are equal. (3) The measure of each of the angles is 60°. (4) The bisector of each angle is perpendicular bisector of the opposite side.
21. A right-angled triangle.	(1) One angle is a right angle (from definition). (2) The sum of other two angles is equal to one right

Given figure or figures	The properties we get
	one right angle. (3) Hypotenuse is the greatest side. (4) Other two sides are perpendicular to each other. (5) The square on the hypotenuse is equal to the sum of the squares on other two sides. (6) The semi-circle drawn with the hypotenuse as diameter passes through the right angular point.
22. The square on the greatest side of a triangle is equal to the sum of the squares on the other two sides.	The triangle is right-angled and the angle opposite to the greatest side is a right angle.
23. An obtuse-angled triangle.	(1) One angle is obtuse (from definition). (2) Each of the other two angles is acute. (3) The side opposite to the obtuse angle is the greatest.
24. An acute-angled triangle.	Each angle is acute (from definition).
25. Two triangles are congruent.	(1) Corresponding sides are equal to each other. (2) Corresponding angles are equal to each other. (3) The two triangles are equal in area.

Given figure or figures	The properties we get
26. The mid-points of two sides of a triangle are joined.	(1) The straight line that joins the mid-points is parallel to the third side. (2) The line segment that joins the mid-points is half the third side.
27. Through the mid-point of a side of a triangle a straight line parallel to another side is drawn.	(1) The parallel straight line bisects the third side. (2) The intercept from the parallel straight line by the sides of the triangle is half the second side.
28. Two triangles stand on the same base and between the same parallels.	The two triangles are equal in area.
29. Two triangles having equal area stand on the same base and on the same side of it.	The straight line that joins the vertices is parallel to the base.
30. The perpendicular bisectors of two sides of a triangle intersect each other.	(1) The distances of the vertices from the point of intersection are equal. (2) The perpendicular drawn from the point of intersection to the third side bisects the third side. (3) The straight line that joins the point of intersection and the mid-point of the third side is perpendicular to the third side.

Given figure or figures	The properties we get
31. The bisectors of two angles of a triangle intersect each other.	(1) The perpendicular length of the sides of the triangle from the point of intersection are equal. (2) The straight line that joins the point of intersection and the third vertex bisects the third angle.
32. Perpendiculars are drawn from the vertices of a triangle to the opposite sides.	(1) Perpendiculars are concurrent. (2) The perperdiculars are the bisectors of the angles of the triangle formed by joining the feet of the perpendiculars (that is, the pedal triangle). (3) The perpendiculars are the perpendicular bisectors of the sides of the triangle formed by the straight lines through the vertices parallel to the opposite sides.
33. In a triangle a median is drawn.	(1) The median bisects the side opposite to the vertex through which the median is drawn. (2) The median divides the triangle into two triangles having equal area.

Given figure or figures	The properties we get
34. Two medians of a triangle intersect each other.	(1) The line segment that joins the point of intersection and the third vertex, when produced, bisects the third side; that is, that line segment, when produced, is the third median. (2) Any median is divided at the point of intersection in the ratio 2 : 1. (3) The triangles formed by joining the point of intersection with the vertices are equal in area.
35. Two triangles have the same vertex and their bases are on the same straight line.	(1) The areas of the triangles bear the same ratio to the bases. (2) Triangles have the same altitude.
36. The bases of two triangles are on the same straight line and their vertices are on a straight line parallel to the former one.	(1) The altitudes of two triangles are equal. (2) The areas of the triangles bear the same ratio to the bases.
37. A straight line parallel to a side of a triangle intersects the other two sides.	The intercepts of the other two sides bear the same ratio.
38. A straight line divides two sides of a triangle proportionally.	The straight line is parallel to the third side.

Given figure or figures	The properties we get
39. Two angles of one triangle are equal to two angles of another.	(1) The third angles of the two triangles are also equal. (2) Two triangles are equiangular.
40. Two triangles are equiangular.	(1) Corresponding sides of two triangles are proportional. (2) Two triangles are similar.
41. The sides of two triangles are proportional.	(1) Two triangles are equiangular. (2) Two triangles are similar.
42. Two sides of a triangle bear the same ratio to the two sides of another and angles included by those two sides are equal.	(1) The two triangles are equiangular. (2) Corresponding sides of two triangles are proportional. (3) Two triangles are similar.
43. Perpendicular is drawn to the hypotenuse of a right-angled triangle from the right angular point.	The two triangles so formed are similar to the original triangle and they are similar to each other.
44. A quadrilateral.	The sum of four angles is equal to four right angles.

Given figure or figures	The properties we get
45. A trapezium.	(1) Two opposite sides are parallel (from definition). (2) The sum of the angles adjacent to any oblique side is equal to two right angles.
46. The mid-points of two oblique sides of a trapezium are joined.	(1) The line segment that joins the mid-points is parallel to the parallel sides. (2) The length of the line segment that joins the mid-points is half the sum of the lengths of two parallel sides.
47. Ar isosceles trapezium.	(1) Two opposite sides are parallel (from definition). (2) Two oblique sides are equal (from definition). (3) Two angles adjacent to any one of the parallel sides are equal to each other. (4) Two angles adjacent to any of the oblique sides are together equal to two right angles. (5) Two diagonals are equal to each other and the distances of the end points of any one of the parallel sides from the point of intersection of the diagonals are equal. (6) The vertices are concyclic.

Given figure or figures	The properties we get
48. A parallelogram.	(1) Two opposite sides are parallel (from definition). (2) Two opposite sides are equal to each other. (3) Two opposite angles are equal to each other.
	(4) Two diagonals bisect each other. (5) Any diagonal divides the parallelogram into two congruent triangles. (6) The two triangles formed by any side at two other vertices are equal in area.
49. In a quadrilateral *(i)* each pair of opposite sides are parallel to each other, or *(ii)* each pair of opposite sides are equal to each other, or *(iii)* each pair of opposite angles are equal to each other, or *(iv)* two opposite sides are parallel and equal to each other, or *(v)* twodiagonal bisect each other, or *(vi)* each diagonal divides the quadrilateral into two congruent triangles.	The quadrilateral is a parallelogram.

Given figure or figures	The properties we get
50. Two parallelograms stand on the same base and between the same parallels.	Two parallelograms are equal in area.
51. A parallelogram and a triangle stand on the same base and between the same parallels.	The area of the triangle is half that of the parallelogram.
52. A rectangle.	(1) It is a parallelogram; hence it holds all the properties of a parallelogram. (2) The measure of each angle is one right angle. (3) The two diagonals are equal to each other. (4) The vertices are concyclic; any diagonal is diameter of that circle and the point of intersection of the diagonals is the centre of that circle.
53. A quadrilateral holds *(i)* any one of the properties of a parallelogram and any one of its angles is a right angle, or *(ii)* any one of the properties of a parallelogram and two diagonals are equal.	The quadrilateral is a rectangle.

Given figure or figures	The properties we get
54. A square.	(1) It is a rectangle; hence it holds all the properties of a rectangle. (2) Four sides are equal to each other. (3) Two diagonals are equal and one bisects the other at right angles. (4) Any diagonal bisects two angles formed at the corresponding vertices; that is, any diagonal divides the square into two right-angled isosceles triangles.
55. A quadrilateral holds *(i)* any one of the properties of a rectangle and two diagonals bisect each other at right angles, or *(ii)* any one of the properties of a rectangle and two adjacent sides are equal.	The quadrilateral is a square.
56. A rhombus.	(1) Four sides are equal to each other. (2) No angle is a right angle. (3) Diagonals are unequal and one bisects the other at right angles. (4) Any diagonal bisects two angles formed at the corresponding vertices. (5) It is a parallelogram,

Given figure or figures	The properties we get
	hence it holds all the properties of a parallellogram. (6) The vertices are not concyclic.
57. In a quadrilateral *(i)* four sides are equal but no angle is a right angle, or *(ii)* two diagonals are unequal and they bisect each other at right angles, or *(iii)* the quadrilateral holds any one of the properties of a parallelogram, and two adjacent sides are equal but no angle is a right angle.	The quadrilateral is a rhombus.
58. A circle.	(1) All diameters are equal. (2) All radii are equal. (3) A radius is half a diameter. (4) The mid-point of a diameter is the centre of the circle.
59. Two equal chords of a circle.	(1) Two chords subten equal angles at the centre. (2) Corresponding arcs intercepted by the chords are equal. (3) Two chords are equidistant from the centre.

Given figure or figures	The properties we get
60. Two equal arcs of a circle.	(1) Two arcs subtend equal angles at the centre. (2) Two chords that join the end points of each arc are equal. (3) Two angles at the circumference subtended by the arcs are equal to each other.
61. A chord of a circle and perpendicular to that chord from the centre.	The perpendicular bisects the chord.
62. A chord of a circle and the straight line that joins the centre and the mid-point of the chord.	The straight line that joins the centre and the mid-point of the chord is perpendicular to the chord.
63. The perpendicular bisectors of two non-parallel chords of a circle.	The point of intersection of two perpendicular bisectors is the centre of the circle.
64. The perpendicular bisectors of two parallel chords of a circle.	A straight line passing through the centre of the circle.
65. An angle at the circumference and the angle at the centre standing on the same arc of a circle.	The angle at the centre is twice the angle at the circumference.

Given figure or figures	The properties we get
66. Two angles in the same segment of a circle, or, two angles at the circumference standing on the same arc.	Two angles are equal to each other.
67. An angle in the semi-circle.	The measure of the angle is one right angle.
68. An angle in the segment of a circle greater than a semi-circle.	The angle is acute.
69. An angle in the segment of a circle lesser than a semi-circle.	The angle is obtuse.
70. A cyclic quadrilateral.	The opposite angles are supplementary.
71. A line segment subtends two equal angles at two points on the same side of it.	Two end points of the line segment and vertices of two angles are concyclic.
72. The sum of two opposite angles of a quadrilateral is equal to two right angles.	The quadrilateral is a cyclic one; that is, the vertices of the quadrilateral are concyclic.
73. The tangent to the circle at a point on it.	All points on the tang excepting the point of co tact are outside the circle.
74. A tangent to a circle and the radius passing	The tangent and the radius through the point of con-

Given figure or figures	The properties we get
through the point of contact	tact are perpendicular to each other.
75. Perpendicular to the tangent to a circle at the point of contact.	The perpendicular passes through the centre of the circle.
76. Perpendicular to the radius of a circle at the point where the radius meets the circle.	The perpendicular is tangent to the circle at that point.
77. Two tangents to a circle from an external point.	(1) The two tangents are equal in length. (2) The two tangents subtend equal angles at the centre. (3) The straight line that joins the centre and the external point is the bisector of the angle contained by the tangents. (4) That straight line bisects the chord of contact at right angles.
78. Two circles touch each other externally.	(1) Centres of two circles and the point of contact are collinear. (2) The distance between the two centres is equal to the sum of the two radii.
79. Two circles touch each other internally.	(1) Centres of two circles and the point of contact are collinear. (2) The distance between the two centres is equal to the difference of two radii.

Second Table

To prove	To be given for proof
1. An angle is a right angle.	(1) If a line segment stands on a straight line, the adjacent angles so formed are equal. (2) A straight line is perpendicular to another. (3) An angle of a triangle is equal to the sum of other two angles. (4) The angle is opposite to the hypotenuse of a right-angled triangle. (5) The angle is any angle of a rectangle or a square. (6) The angle is contained by the two diagonals of a rhombus or those of a square. (7) The angle is an angle in the semi-circle. (8) The angle is contained by a tangent and the radius through the point of contact. (9) The angle is contained by a chord and the line segment that joins the mid-point of the chord and the centre of the circle. (10) The angle is contained by the internal and external bisectors of an angle. (11) The angle formed by

To prove	To be given for proof
	the base with the bisector of the vertical angle of an isosceles triangle or the median drawn from the vertex to the base. (12) The angle is such an angle of a triangle, the median drawn from the vertex of which is half the third side.
2. Two line segments are on a straight line or three points are collinear	(1) The measure of the angle contained by those two line segments is 180° (2) The straight line drawn through the common point of those two line segments makes equal angles on opposite sides.
	(3) Those two line segments are perpendiculars to a straight line drawn through the common point of the line segments. (4) If those two line segments have a common point and each of them is parallel to a straight line. (5) Two of the three points are joined with the third point, particularly, the point between these two points and if any of the above mentioned properties are obtained. (6) If the three points are two opposite vertices of a

To prove	To be given for proof
	parallelogram and the mid-point of the diagonal that does not pass through those two vertices. (7) If the three points are the centre of a circle and mid-points of two parallel chords. (8) If the three points are the centroid of a triangle, a vertex and the mid-point of the opposite side. (9) If the three points are the points of contact of two parallel tangents to a circle and the centre of the circle. (10) If the three points are the points of contact of a tangent to a circle, mid-points of a chord parallel to the tangent and the centre of the circle. (11) If the three points are the point of contact of two circles touching each other and the centres of those two circles.
3. Two angles are equal.	(1) Each of the two angles is a right angle. (2) If a straight line is perpendicular to another straight line. (3) Two angles are vertically opposite angles. (4) If the two angles are corresponding angles or

To prove	To be given for proof
	alternate angles formed by intersection of each of two parallel straight lines by another straight line.
	(5) If the angles are base angles of an isosceles triangle.
	(6) Two parallel straight lines make the angle with two other parallel straight lines in the same direction.
	(7) Two angles are corresponding angles of two congruent triangles.
	(8) If the two angles are angles of two triangles of which the sum of other two angles of the triangles are equal.
	(9) Two angles are opposite angles of a parallelogram.
	(10) Two angles are adjacent with respect to any of the parallel sides of an isosceles trapezium.
	(11) The angles are equal to half or twice the same angle or equal angles.
	(12) Two angles are in the same segment of a circle or angles at the circumference subtened by the same arc.
	(13) Two angles are subtended at the centre of a circle by two equal chords, or subtended by two equal arcs at a point or two points

To prove	To be given for proof
	on the third arc. (14) One of the two angles is an angle of a cyclic quadrilateral and the other is the corresponding external angle. (15) Two angles are subtended at the centre of a circle by two tangents drawn from a point outside the circle. (16) Two corresponding angles of two triangles of which sides are proportional.
4. Two line segments are equal.	(1) Two line segments are the sides of a triangle opposite to two equal angles. (2) A triangle is isosceles or equilateral. (3) The angles adjacent to any of the parallel sides of a trapezium are equal. (4) The oblique sides of an isosceles trapezium. (5) Two line segments are corresponding sides of two congruent triangles. (6) Two line segments are opposite sides of a parallelogram. (7) Two line segments are diagonals of a rectangle. (8) Two line segments are any two sides of a rhombus

To prove	To be given for proof
	or a square.
	(9) Two line segments are equal to half or twice the same line segment or equal line segments.
	(10) Two line segments are the bisectors of base angles of an isosceles triangle or the perpendiculars or medians drawn through those two vertices to the opposite sides.
	(11) Two line segments are diameters or radii of a circle.
	(12) Two line segments are two portions of a chord which are obtained by dropping perpendicular on the chord from the centre.
	(13) One of the two line segments is half the hypotenuse of a right-angled triangle and the other is the median on the hypotenuse.
	(14) Two line segments are the distances of two end points of a line segment from any point on the perpendicular bisector of that line segment.
	(15) Two line segments are the distances of two intersecting straight lines from any point on the bisector of the angle in-

To prove	To be given for proof
	cluded by those two straight lines. (16) Two line segments are the tangents to a circle from a point outside the circle. (17) Two line segments are two chords of a circle equidistant from the centre.
5. Two triangles are congruent.	(1) Two sides and the angles included by them are equal. (2) Two angles and a corresponding side are equal. (3) Three sides are equal. (4) In two right-angled triangles, hypotenuse and another side are equal.
6. Two straight lines or two line segments are parallel.	(1) When the two straight lines are intersected by another straight line, two corresponding angles are equal or two alternate angles are equal or two interior angles on the same side of the transversal are together equal to two right angles. (2) Two straight lines are perpendicular to another straight line. (3) Each of the two straight lines is parallel to another straight line.

To prove	To be given for proof
	(4) Two line segments are the opposite sides of a parallelogram.
	(5) One of the two line segments is a side of a triangle and the other is the line segment that joins the mid-points of other two sides.
	(6) One of the two line segments is the common side of two triangles having equal area and lying on the same side of the common side and the corresponding vertices are on the other line segment.
	(7) Two line segments are the opposite sides of a trapezium (not any two opposite sides.)
	(8) Two straight lines are the tangents to a circle at two end points of a diameter.
	(9) One of the line segments is a side of a triangle and the other is the line segment that joins the points at which the other two sides are divided proportionally.
	(10) Arms of two equal angles fromed in the same direction at two points on two parallel straight lines.

To prove	To be given for proof
7. A straight line is perpendicular to another straight line.	All the conditions by which an angle can be proved to be a right angle are applicable. The two arms of a right-angle are perpendicular to each other. Moreover, there is another condition: If from one point on a straight line, line segments are drawn to another straight line and if the line segment which is the part of the first line is the least.
8. A quadrilateral is a parallelogram.	(1) Opposite sides are parallel. (2) Opposite sides are equal. (3) Opposite angles are equal. (4) Any two opposite sides are parallel and equal. (5) Any two opposite sides are parallel and any two opposite angles are equal. (6) Two diagonals bisect each other. (7) Each diagonal divides the quadrilateral into two congruent triangles.
9. A quadrilateral is a rectangle.	(1) The quadrilateral is a parallelogram and any angle is a right angle. (2) The quadrilateral is a parallelogram and two dia-

To prove	To be given for proof
	gonals are equal. (3) The quadrilateral is a cyclic parallelogram.
10. A quadrilateral is a square.	(1) The quadrilateral is a rectangle and any two adjacent sides are equal. (2) The quadrilateral is a rectangle and the diagonals are at right angles. (3) The quadrilateral is a rectangle and a diagonal bisects the opposite angles. (4) The quadrilateral is a cyclic one and its diagonals are diameters of the circle which are at right angles.
11. A quadrilateral is a rhombus.	(1) Four sides of the quadrilateral are equal, but no angle is a right angle. (2) The quadrilateral is a parallelogram (not a rectangle) and any two adjacent sides are equal. (3) The diagonals of the quadrilateral are unequal and the diagonals bisect each other at right angles.
12. A quadrilateral is an isosceles trapezium.	(1) Two opposite sides of quadrilateral are parallel and the other two sides (oblique sides) are equal. (2) Two opposite sides of the quadrilateral are para-

To prove	To be given for proof
	llel and two angles adjacent to any of the parallel sides are equal. (3) The quadrilateral is a cyclic trapezium.
13. Two parallelograms are equal in area.	Two parallelograms stand on the same base or equal bases and within the same parallels or having same altitude.
14. Two triangles are equal in area.	(1) Two triangles are congruent. (2) Two triangles stand on the same base or equal bases and within the same parallels or having same altitude. (3) Two triangles have the same base as that of a parallelogram and they are within the same parallels.
15. The areas of two triangles bear the same ratio as that of their bases.	The bases of two triangles are on the same straight line and they have the same vertex or their vertices are on a straight line parallel to their bases.
16. A triangle is isosceles.	(1) Two sides of the triangle are equal. (2) Two angles of the triangle are equal.

To prove	To be given for proof
	(3) The bisector of an angle of the triangle bisects the base at right angles.
17. A triangle is equilateral.	(1) The measure of each of the two angles of the triangle is 60°. (2) All sides of the triangle are equal. (3) The bisectors of any two angles of the triangle bisect the corresponding opposite sides at right angles.
18. A triangle is right-angled.	(1) One angle of the triangle is a right angle. (2) The sum of two angles of the triangle is one right angle or equal to the third angle. (3) The triangle is inscribed in a circle and one side of the triangle is a diameter. (4) The square on the greatest side of the triangle is equal to sum of the squares on other two sides.
19. Two triangles are equiangular.	(1) Any two angles of a triangle are equal to two angles of the other. (2) The sides of two triangles are proportional.
20. Two triangles are similar.	(1) Two triangles are equiangular. (2) The sides of two triang-

To prove	To be given for proof
	les are proportional. (3) The ratio of two sides of a triangle is equal to the ratio of two sides of the other and the angles contained by those two sides are equal. (4) If two triangles are obtained by dropping perpendicular to the hypotenuse of a right-angled triangle from the right angular point, or, one of the two triangles is the triangle obtained as mentioned above and the other is the original triangle.
21. Four points are concyclic.	(1) If any two of those four points are on the same side of the line segment that joins the other two points and the line segment subtends equal angles at the former two points. (2) The four points are vertices of a quadrilateral, the two opposite angles of which are supplementary.
22. A straight line passes through the centre of a circle.	(1) the straight line is perpendicular bisector of a chord of the circle. (2) The straight line is the bisector of the angle contained by two tangents to the circle.

To prove	To be given for proof
23. A straight line is a tangent to a circle.	(1) The straight line is perpendicular to a radius of the circle at the point where the radius meets the circle.

[*Remark:* The informations enlisted in these two tables may not be exhaustive. Hence, any addition may be made to these tables, which are logical.]

MENSURATION

CHAPTER 1

Triangle

In Class VII you have learnt to find the perimeter and the area of rectangles and squares. Besides rectangles and squares we come across figures and articles of other different shapes in our daily life. Examples of some of those are given in Fig. 1.

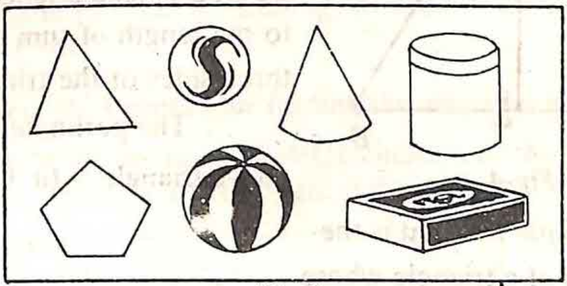


Fig-1

At different times we are required to determine the length, breadth, height, area, volume etc. of these figures and articles. The practical art of measuring the length, breadth, perimeter, area and volume of figures and articles is known as mensuration in Mathematics.

In Class VII you have learnt the following :
Perimeter of a rectangle = 2 (length + breadth)
Area of a rectangle = length × breadth

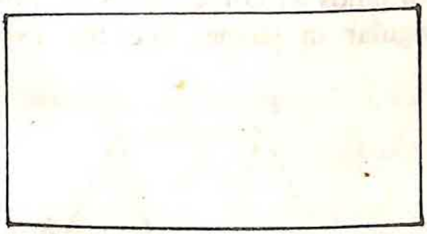


Fig-2

Fig-3

Premiter of a square = 4 × length of one side
Area of a square = (length of one side)^2

You will now learn how the perimeter and the area of triangles and circles are determined.

Perimeter of triangles :

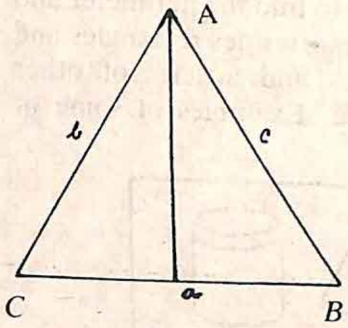


Fig-4

If ABC be a triangular field with sides a, b and c, then to enclose it by a boundary, we would need a fence equal to the length of sum of the three sides of the triangle.

$\therefore$ The perimeter of the above triangle $= (a + b + c)$

Example 1: What is the perimeter of a triangle whose sides are 15 metres, 18 metres and 24 metres respectively?

Perimeter of the triangle

$= (15 + 18 + 24)$ metres

$= 57$ metres.

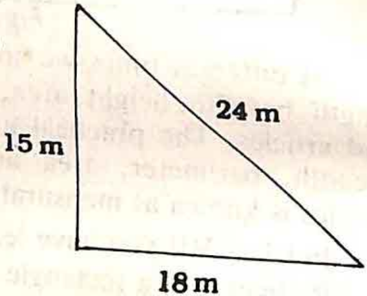


Fig-5

Area of triangles :

You know how to find the area of a rectangle and a square. But, all the fields or lands we come across are not actually square or rectangular in shape. See the two diagrams below:

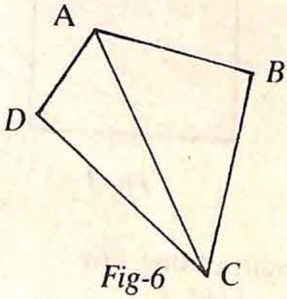


Fig-6

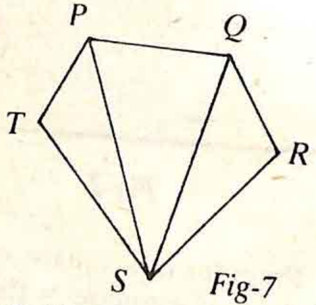


Fig-7

The quadrilateral $ABCD$ is neither a rectangle nor a square. Again figure *PQRST* is a pentagon.

Therefore it is not possible to find the area of these figures with the help of the rules applicable to rectangle and square. But if you notice carefully you will find that they can be divided into more than one triangle. So if we can find the area of these triangles individually, then we will be able to find the area of the whole figure. So it is also necessary to find the area of triangles individually. Therefore, the formula for finding the area of a triangle is very important.

Deduction of the formula for finding the area of a triangle:

According to the formula for determination of the area of a rectangle, the area of rectangle $ABCD = DC \times AD$

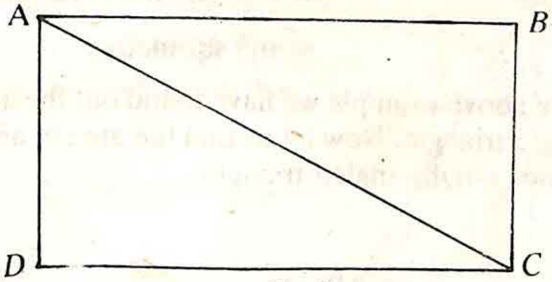


Fig-8

You also know that the diagonal of a rectangle divides it into two equal parts. In the above figure diagonal AC divides the rectangle $ABCD$ into two triangles $\triangle ABC$ and $\triangle ADC$.

$\therefore$ We can write

$$\begin{aligned}\text{area of } \triangle ADC &= \text{area of } \triangle ABC\\ &= \tfrac{1}{2} \text{ area of rectangle } ABCD\\ &= \tfrac{1}{2}\, DC \times AD\\ &= \tfrac{1}{2} \text{ base} \times \text{altitude}\end{aligned}$$

$\because$ in the $\triangle ADC$, DC = base and AD = altitude,

so it can be said, area of the triangle = $\frac{1}{2}$ base × altitude

[*Remark:* observe that in this case, both the triangles are right-angled and if one of the perpendicular sides be taken as base, the other is the altitude.]

Example 2. 18 m and 12 m are the length and breadth respectively of a rectangle $ABCD$. What is the area of $\triangle ABC$ or $\triangle ADC$?

Area of rectangle = (18×12) sqq. meters = 216 sq. meters

$\therefore$ Now area of $\triangle ABC$ or $\triangle ADC$

$= (216 \div 2)$ sq. metres

$= 108$ sq. metres

Or, Area of $\triangle ABC$ = Area of $\triangle ADC$.

$= \frac{1}{2}$ base × altitude

$= \frac{1}{2} \times (18 \times 12)$ sq. metres

$= 108$ sq. metres

In the above example we have found out the area of a right-angled triangle. Now let us find the area of a triangle which is not a right-angled triangle.

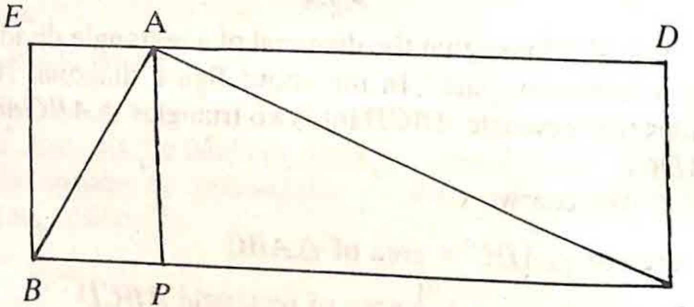


Fig-9

In the figure above, ABC is a scalene triangle.

Through the vertex A, ED is drawn parallel to BC. BE and CD are perpendiculars on BC. Thus we get $BCDE$ as the rectangle with base BC and altitude BE. Again $BE = AP$ = altitude of $\triangle ABC$.

You know area of $\triangle ABC$

$= \frac{1}{2}$ area of the rectangle $BCDE$

$= \frac{1}{2}\ BC.BE$

$= \frac{1}{2}$ BC.AP

$= \frac{1}{2}$ base × altitude

$\therefore$ we can say that for any triangle

area of the triangle = $\frac{1}{2}$ base × altitude

Area of an equilateral triangle :

ABC is an equilateral triangle, each sides of which is equal and AD is the median which indicates the altitude of the triangle.

Let the length of each side of $\triangle ABC = a$

$\therefore\ AB = a$ units and

$BD = \frac{1}{2}\ BC = \frac{a}{2}$ units.

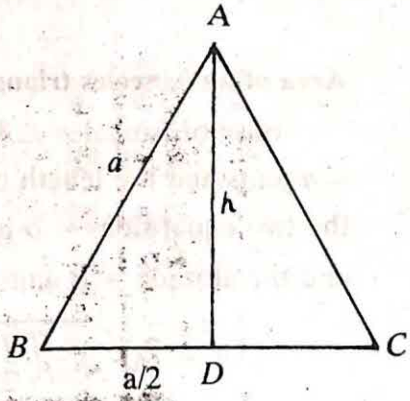


Fig-10

You know $AD^2 = AB^2 - BD^2$.

$$\therefore\ AD = \sqrt{AB^2 - BD^2} = \sqrt{a^2 - \left(\frac{a}{2}\right)^2} \text{ units}$$

$$= \sqrt{\frac{3a^2}{4}} \text{ units} = \frac{\sqrt{3}}{2}\, a \text{ units}$$

Altitude of an equilateral triangle = $\frac{\sqrt{3}}{2}$ × length of its side

∴ Area of the equilateral triangle ABC

$$= \frac{1}{2} \text{ base} \times \text{altitude}$$

$$= \frac{1}{2} a \times \frac{\sqrt{3}}{2} a \text{ sq. unit} = \frac{\sqrt{3}}{4} a^2 \text{ sq. unit.}$$

Area of an equilateral triangle $= \frac{\sqrt{3}}{4}(\text{side})^2$

Example 3. Find the area of an equilateral triangle whose sides are 4 cm.

Area of the triangle $= \frac{\sqrt{3}}{4} \times 4^2$ sq. cm.

$= 4\sqrt{3}$ sq. cm.

Area of an isosceles triangle :

Base of isosceles $\triangle ABC$ $= a$ units and the length of the two equal sides $= b$ units and the altitude $= h$ units.

$$\therefore h = \sqrt{b^2 - \left(\frac{a}{2}\right)^2}$$

$$= \sqrt{b^2 - \frac{a^2}{4}}$$

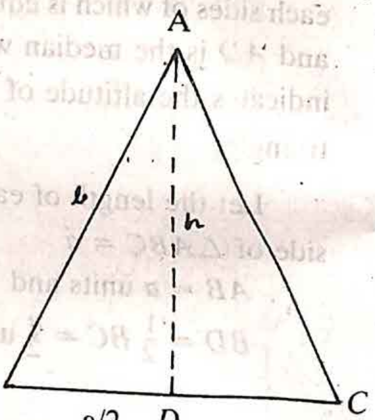


Fig-11

Area of isosceles triangle ABC $= \frac{1}{2} \times a \times \sqrt{b^2 - \frac{a^2}{4}}$ sq. unit.

Example 4. The base of an isosceles triangle is 8 cm, and the other two sides are of length 10 cm each. Find the area of the triangle.

The required area of the triangle,

$$= \frac{1}{2} \times 8 \times \sqrt{(10)^2 - \frac{8^2}{4}} \text{ sq.cm} = 8\sqrt{21} \text{ sq. cm.}$$

To find the area of a triangle when the lengths of the three sides are known :

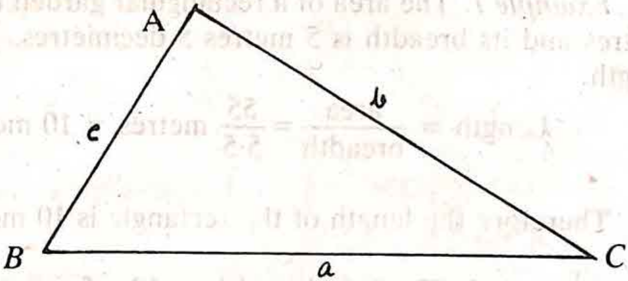


Fig-12

The lengths of the three sides of the triangle ABC are a, b and c units respectively,

$\therefore$ the perimeter of the triangle = $(a + b + c)$ units.

$\therefore$ semi-perimeter $= s = \dfrac{a+b+c}{2}$ units

Then the area of the triangle will be

$$= \sqrt{s(s-a)\,(s-b)\,(s-c)} \text{ Sq. units.}$$

Example 5. Find the area of a triangle whose sides are 26 cm, 28 cm and 30 cm respectively.

$$s = \frac{26+28+30}{2} \text{ cm} = \frac{84}{2} \text{ cm} = 42 \text{ cm.}$$

$\therefore$ Area of the triangle

$$= \sqrt{42(42-26)(42-28)(42-30)} \text{ sq. cm.}$$

$$= \sqrt{42\times16\times14\times12} \text{ sq. cm.}$$

$$= \sqrt{7\times6\times6\times2\times2\times7\times4\times4} \text{ sq. cm.}$$

$$= 7\times6\times4\times2 \text{ sq. cm.}$$

$$= 336 \text{ sq. cm.}$$

Miscellaneous Examples :

Example 1. The area of a rectangular garden is 55 sq. metres and its breadth is 5 metres 5 decimetres. Find its length.

$$\text{Length} = \frac{\text{area}}{\text{breadth}} = \frac{55}{5{\cdot}5} \text{ metres} = 10 \text{ metres}$$

Therefore the length of the rectangle is 10 metres.

Example 2. The length and breadth of a rectangular garden are 100 metres and 80 metres respectively. A path 5 metres wide runs within it along the sides of the garden. What is the area of the path?

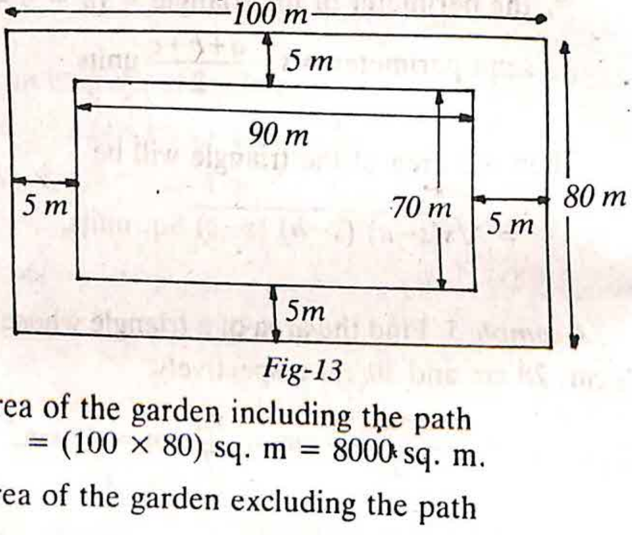


Fig-13

Area of the garden including the path
$= (100 \times 80)$ sq. m $= 8000$ sq. m.

Area of the garden excluding the path

$= [\{100 - (5 \times 2)\} \times \{80 - (5 \times 2)\}]$ sq. m.
$= (90 \times 70)$ sq. m.
$= 6300$ sq. m.

$\therefore$ the area of the path $= (8000 - 6300)$ sq. m $= 1700$ sq. m.

Example 3. The diagonals of a parallelogram are 3 cm and 4 cm respectively and they intersect each other at right angles. Find the lengths of the sides of the parallelogram. What is the characteristic of the parallelogram?

We know diagonals of a parallelogram bisect each other

$\therefore AO = OC = 1{\cdot}5$ cm.
$BO = OD = 2$ cm.

$\therefore$ According to the problem,

$\triangle AOB$ is a right-angled triangle.

$AB = \sqrt{2^2 + 1{\cdot}5^2} = \sqrt{6{\cdot}25}$
$= 2{\cdot}5$ cm.

$\therefore AB = DC = 2{\cdot}5$ cm.

Similarly, $BC = AD = 2{\cdot}5$ cm.

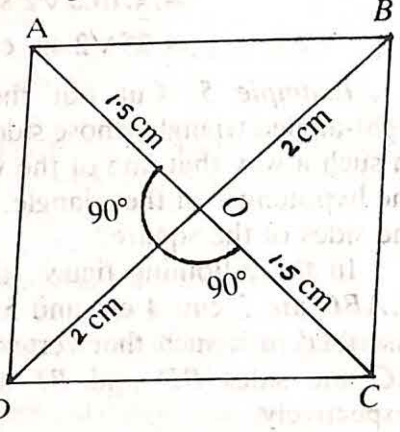


Fig-14

$\therefore$ all the sides of the parallelogram are 2·5 cm long.

The parallelogram is a rhombus.

Example 4. Find the area of an isosceles triangle whose equal sides are 10 cm each and the angle included by two sides is 45°.

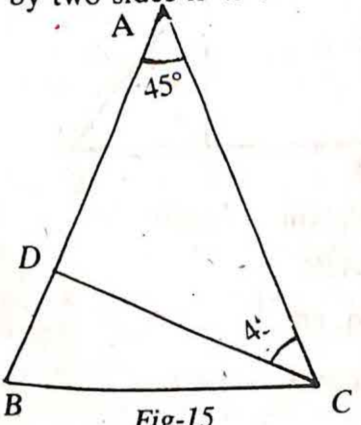


Fig-15

In $\triangle ABC$, $AB = AC = 10$ cm and $\angle A = 45°$.

Perpendicular CD is dropped on AB.

$\therefore$ according to construction, $\angle ACD = 45°$

$\therefore AD = CD$

Again, in right-angled triangle ADC,

$$AD^2 + CD^2 = AC^2 = 10^2 \text{ sq. cm.} = 100 \text{ sq. cm.}$$

$\therefore 2CD^2 = 100$ sq. cm.
$CD^2 = 50$ sq. cm.
$CD = 5\sqrt{2}$ cm.

$\therefore$ Area of the triangle
$= \frac{1}{2} AB.CD$

$= \frac{1}{2}.10.5\sqrt{2}$ sq. cm.

$= 25\sqrt{2}$ sq. cm.

Example 5. Cut out the biggest square from a right-angled triangle whose sides are 3 cm, 4 cm and 5 cm, in such a way that one of the vertices of the square is on the hypotenuse of the triangle. What will be the length of the sides of the square?

In the adjoining figure, the lengths of the sides of $\triangle ABC$ are 3 cm, 4 cm and 5 cm. The square $BDEF$ is inscribed in it such that vertex E lies on the hypotenuse AC and sides BD and BF are on sides BA and BC respectively.

Now, the length of each side of the square is to be found out.

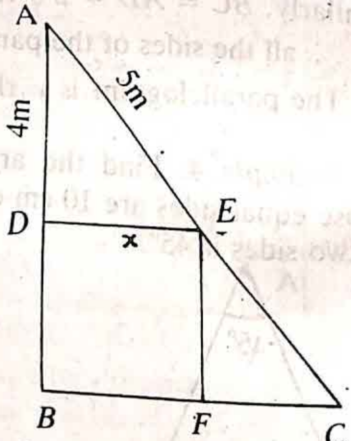


Fig-16

Let the length of the sides of the square be x cm.

$\therefore AD = (4 - x)$ cm,

$CF = (3 - x)$ cm

$\therefore$ the area of $\triangle ADE$

$= \frac{1}{2} \times (4 - x)x$ sq. cm

Again the area of $\triangle EFC$

$= \frac{1}{2} \times (3 - x)x$ sq. cm.

Area of square $BDEF = x^2$ sq. cm.

But you know the area of $\triangle ABC$

$= \frac{1}{2}\, 4 \times 3$ sq. cm. $= 6$ sq. cm.

$\therefore x^2 + \frac{1}{2}(4-x)x + \frac{1}{2}(3-x)x = 6$

or, $x^2 + 2x - \frac{x^2}{2} + \frac{3x}{2} - \frac{x^2}{2} = 6$

or, $\frac{7x}{2} = 6$

$\therefore x = \frac{12}{7} = 1\frac{5}{7}$

$\therefore$ the length of the sides of the required square is $1\frac{5}{7}$ cm.

Exercise 1

1. Find the area and perimeter of a rectangular garden of length 15 metres and breadth 10 metres.

2. The perimeter of a rectangle is 120 metres and the difference between its length and breadth is 12 metres. Find the area of the rectangle.

3. The length of one of the diagonals of a parallelogram is 75 metres and the length of the two adjacent sides are 70 metres and 65 metres respectively. Find the area of the parallelogram.

4. In the front portion of a rectangular plot 16 sq. m space is reserved for gardening and a house is built on the remaining portion. At this the length of the plot reduced by 2 metres and the house is on a square plot. Find the area of the land on which the house is situated.

5. The length and breadth of a rectangular court-yard is 15 m and 12 m respectively. How much will it cost to construct a path within the yard along its sides at the rate of Rs. 2·25 per sq. metre?

6. How many tiles each 20 cm long and 17·5 cm wide will be required to pave a court-yard 10 metres long and 7 metres wide?

7. A rectangular field of 60 metres length and 45 metres breadth is divided into 4 equal plots by raising two paths parallel to its sides. If the paths are 6 decimetres wide, what will be the area of each plot? How much land will the paths occupy?

8. What is the area of a triangle having its sides 14 cm and height 5 cm?

9. What is the perimeter of a triangle having its sides 14 cm, 13 cm and 18 cm respectively?

10. What is the height and area of an equilateral triangle with sides 8 cm long?

11. The height of an equilateral triangle is 15 cm. What will be its perimeter and area?

12. Prove that the numerical value of the area of a right-angled triangle is equal to half of the numerical value of the product of the lengths of the sides holding the right angle and hence find the area of a right-angled triangle in which the sides holding the right angle are 7 cm and 12 cm respectively.

13. In a right-angled triangle the sides holding the right angle are 12 cm and 5 cm respectively. Find the length of the perpendicular from the vertex on the hypotenuse. (approx. upto 3 places of decimal)

14. What are the area and perimeter of an equilateral triangle whose height is $4\sqrt{3}$ cm?

15. The base of an isosceles triangle is 6 cm and the length of each of its equal sides is 5 cm. What is its area?

16. The length of each ot the equal sides of an isosceles triangle is 10 cm and its height is 8 cm. Find the length of its base and area.

17. The three sides of a triangle are 25 cm, 39 cm and 56 cm long. Find the length of the perpendicular on the longest side from the vertex opposite to it.

18. the length of the two equal sides of an isosceles triangle is 15 cm and the length of the bisector of the angle held by the equal sides is 12 cm. Find the perimeter and area of the triangle.

19. The length of the three sides of a triangle are 15 cm, 20 cm and 25 cm respectively. Find the length of the perpendicular on the longest side from the vertex opposite to it.

20. The length of the three sides of a triangle are 10 cm, 14 cm and 16 cm. Find the area of the triangle and from it show that the length of the perpendicular on the longest side from the vertex opposite to it is $5\sqrt{3}$ cm.

21. The ratio of the three sides of a triangular park is 2 : 3 : 4 and its perimeter is 108 metres. Find its area. How much will one have to walk to reach straight to the longest side from the angular point opposite to it?

22. Three villages Satirpara, Narsinghi and Hajipur are not on the same straight line. The respective distances between them are 823 metres, 1250 metres and 1327 metres. How much long a road is to be built to link them in order? How much would cost to build the road at Rs. 9·45 per metre?

23. The upper part of a coconut tree 45 metres ling broke down in storm and its top touched the ground 15 metres away from its foot. What is the area of the triangle formed by joining the foot and top of the tree?

24. Out of an iron sheet 16 dcm long and 12 dcm wide 8 equal pieces have been cut off. The pieces are of right-angled triangular shape such that the sides adjacent to the right angle are half of both the length and width of the rectangle. If no portion of the sheet is waste, what will be the perimeter and area of each piece?

25. If 25 metres long ladder is placed just to touch the top of a 20 metres long chimney, how far from the foot of the chimney will the foot of the ladder rest?

26. The ratio of the three sides of a triangle is 4 : 5 : 7 and its perimeter is 144 metres. Find the area of the triangle.

27. Starting from the same port one of the two ships has gone 60 km per hr southward , while the other has gone 90 km per hr eastward Find the distance between the two ships after one hour.

28. The distance of the bases of two light-post 35 metres and 11 metres high is 10 metres. Find the distance between their tops.

29. The ratio of the heights of two triangles is 3 : 4. If the ratio of their areas is 4 : 3, what will be the ratio of their bases?

30. From a rectangular wooden sheet. the biggest triangular piece that can be cut out has an area 54 sq. cm. If the length of the rectangle be $\frac{4}{3}$ rd of its breadth, what will be the length of its diagonals?

CHAPTER 2

CIRCLE

Circumference of Circle

Take a wire of 44 cm length and bend it in the form of a circle as given in the figure below.

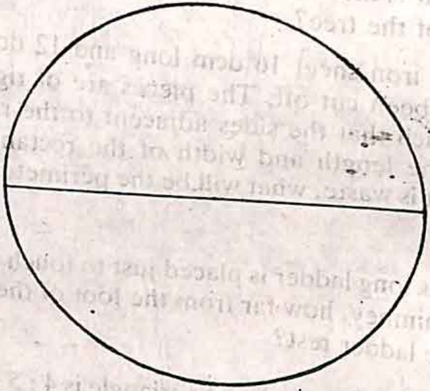


Fig-17

Now measure the diameter of the circle.

Then the circumference of the circle will be 44 cm.

You will find the length of the diameter of the circle is approximately 14 cm.

Therefore the ratio of circumference to diameter will be

circumference : diameter = 44 : 14 (approx)

= 22 : 7 (approx)

Similarly, if a wire of length 66 cm is bent in the form of a circle, you will find its diameter is approxmately 21 cm.

Here also, circumference : diameter = 66 : 21 (approx)

= 22 : 7 (approx)

If circles are formed with wires of different lengths you will find the ratio of circumference to diameter is always approxmately 22 : 7.

So, it can be said that there is a definite and fixed-relation between the circumference of a circle and its diameter, and it is, circumference : diameter = 22 : 7 (approx).

It can be practically verified and can also be verified with the help of geometry.

On a card-board draw three concentric circles of radii 7 cm, 10·5 cm and 14 cm as given in the figure below. Also draw a common diameter which intercepts the three circumferences at *c*, *z* ; *b*, *y* and *a*, *x* points respectively.

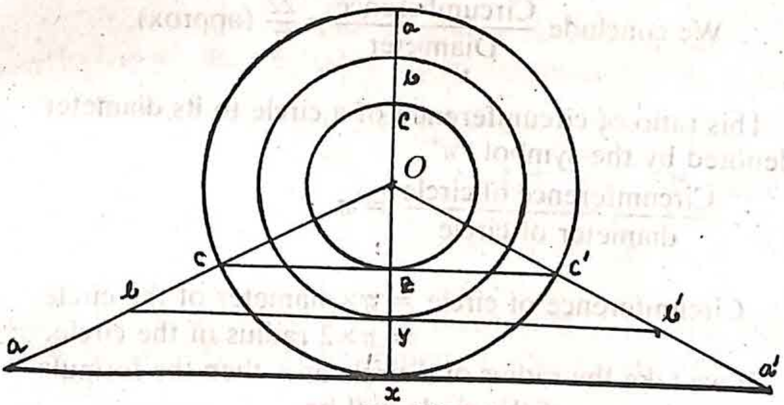


Fig-18

Now, take three pieces of thin wire and by bending them place them in such a way that the open ends meet at the points *c*, *b*, *a* respectively of the common diameter. Now make the pieces of wire straight and place them in such a position that the wires are perperpendicular to the respective diameters at the other end points *z*, *y* and *x* of the respective diameters and the middle points of the wires concide with *z*, *y* and *x* respectively.

Obviously, the three points *a*, *b*, *c* are collinear. If the end points of the wires in their new positions are joined and produced, it will be observed that they lie on the same straight line which passes through the centre, O. If two end points of the wires in their new position be c', c''; b', b'' and a', a'', then measure $a''\,a'$, $b''\ \ b'$ and $c''c'$. You will find that the lengths are approximately 44 cms, 33 cms and 22 cms respectively.

Now, in $\triangle Oc''z$, $\frac{c''z}{Oz} = \frac{22}{7}$

$$\therefore \frac{c''c'}{cz} = \frac{2c''z}{oz} = \frac{\text{Circumference}}{\text{Diameter}} = \frac{22}{7} \text{(approx)}$$

Similarly, we will find from $\triangle Ob''y$ and $Oa''z$ that

$$\frac{\text{Circumference}}{\text{Diameter}} = \frac{22}{7} \text{ (approx)}$$

$$\therefore \text{ We conclude, } \frac{\text{Circumference}}{\text{Diameter}} = \frac{22}{7} \text{ (approx)}$$

This ratio of circumference of a circle to its diameter is denoted by the symbol "π"

$$\therefore \frac{\text{Circumference of circle}}{\text{diameter of circle}} = \pi$$

$$\begin{aligned} \therefore \text{ Circumference of circle} &= \pi \times \text{diameter of the circle} \\ &= \pi \times 2 \text{ radius of the circle.} \end{aligned}$$

If we take the radius of a circle as r, then the formula for circumference of the circle will be

$$\boxed{\text{Circumference of the circle} = 2\pi r \text{ units.}}$$

Corollary: $r = \frac{\text{circumference of circle}}{2\pi}$ units

N.B.: Note that the value of π is usually taken as $\frac{22}{7}$.

It is not possible to determine the value of π exactly.

Example 1. The diameter of a circular park is 35 metres. How many metres of railing is required to put a fence around the park?

You know that the length of railing required for the fence will be equal to the perimeter of the park.

Perimeter of the park $= 2\pi \times r$

$$= 2 \times \frac{22}{7} \times 17{\cdot}5 \text{ metres}$$

$$= 110 \text{ metres}$$

∴ 110 metres of railing is required to put a fence around the park.

Example 2. The diameter of the wheel of a bullock-cart is 1·05 metres. How many revolutions will the wheel make to cover a distance of 495 metres

You know, in one complete revolution the wheel covers a distance equal to the circumference of the wheel.

Circumferece of th^ wheel $= 2\pi \times {\cdot}525$ metres

$$= 2 \times \frac{22}{7} \times {\cdot}525 \text{ metres}$$

$$= 3{\cdot}3 \text{ metres.}$$

Therefore, to cover a distance of 495 metres the wheel will make (495÷3·3) revolutions = 150 revolutions.

Example 3. The radius of the wheel of a train is 0·35 metre. Find the speed of the train if the wheel makes 450 revolutions per minute.

The circumference of the wheel of the train

$$= 2\pi \times 0{\cdot}35 \text{ metres}$$

$$= 2 \times \frac{22}{7} \times 0{\cdot}35 \text{ metres}$$

$$= 2{\cdot}2 \text{ metres}$$

∴ In 1 minute the train travels (2·2 × 450) metres = 990 metres.

∴ In 1 hour or 60 minutes the train travels (990 × 60) metres = 59400 metres.

Therefore, the speed of the train is 59·4 km/hr.

Example 4. There is a square field of sides 4·2 metres. A circular well with the widest opening is dug on this field. How much railing is required to put a boundry around this well?

From the adjoined figure you can find that the diameter of the widest well will be equal to the side of the square *i.e.* 4·2 metres.

$\therefore$ the perimeter of the well

$= 2\pi \times 2{\cdot}1$ metres

$= 2 \times \frac{22}{7} \times 2{\cdot}1$ metres

$= 13{\cdot}2$ metres

$\therefore$ 13·2 metres of railing will be required to put a boundary around the well.

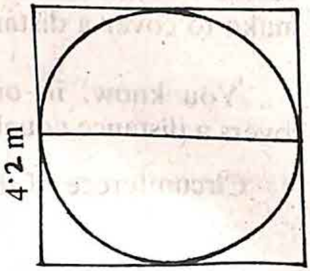


Fig-19

Example 5. The radius of the front wheel of a steam engine is 35 cm and that of the rear wheel 85 cm. How many times will the rear wheel revolve when the front wheel revolves 600 times?

Circumference of the front wheel $= 2 \times \frac{22}{7} \times 17{\cdot}5$ cm

Circumference of the rear wheel $= 2 \times \frac{22}{7} \times 42$ cm

While the front wheel revolves 600 times it covers $2 \times \pi \times 17{\cdot}5 \times 600$ cm

Therefore, to travel that distance, the rear wheel will revolve

$$\frac{2 \times \pi \times 17{\cdot}5 \times 600}{2 \times \pi \times 42} = 250 \text{ times}$$

Answer: The rear wheel will revolve 250 times.

Exercise 2

1. The diameter of a circular park is 42 m. How many metres of railing will be required to put a fence around it?

2. How long wire will be required to make a circular ring of 35 cm diameter?

3. How much will it cost to put a bamboo fence around a circular area of 4 m radius at the rate of Rs. 4·55 per metre?

5. How many revolutions will be required to complete 440 m and 1100 m races respectively in a circular track of radius 35 m?

6. The difference of length of the circumference and diameter of a wheel is 75 cm. What is the radius of that wheel?

7. In a race Ram and Shyam starts from the same point on a circular track of 49 m diameter. Taking 10 complete revolutions while Ram has finished the race Shyam is one revolution behind. How long is the race and by how many metres has Ram beaten Shyam?

8. The radius of the wheels of a train is 0·245 m. If the wheels revolve 500 times per minute, find the speed of the train.

9. While a rope runs over a pulley it revolves 115 times per minute. If the speed of the rope be 2530 cm per minute, what is the radius of the pulley?

10. When a wire is bent to form a circle, the diameter of the circle becomes 5·6 dcm. If a square is formed with that wire, what will be the length of its sides?

11. A square of sides 33 cm long can be formed by a wire. What will be the radius of the circle that can be formed by that wire?

12. The ratio of the circumferences of two circles is 2 : 3 and the difference of the length of their radii is 2 cm. What are the lengths of their diametres?

13. To put a fence around a semi-circular park we require a railing 162 m long. What is the length of the park along its diameter?

14. The internal and external circumferences of a metal ring are 132 cm and 154 cm respectively. Find the width of the ring.

15. How long railing will be required to enclose the maximum circular area within a square field of sides 24·5 m long? If the field is enclosed as it is, how much more railing will be needed?

16. Find the circumference of each of the four maximum sized circular plates that can be had from a brass plate of square size having the area of 196 sq. cm.

17. The wheel of a machine is attached to the wheel of a motor with a belting. The diameters of the wheel of the motor and that of the machine are 14 cm and 94·5 cm respectively. If the wheel of the motor revolves 27 times in one second, how many times will the wheel of the machine revolve per hour?

Area of circle

Draw a polygon with large number of sides around a circle as given in the figure.

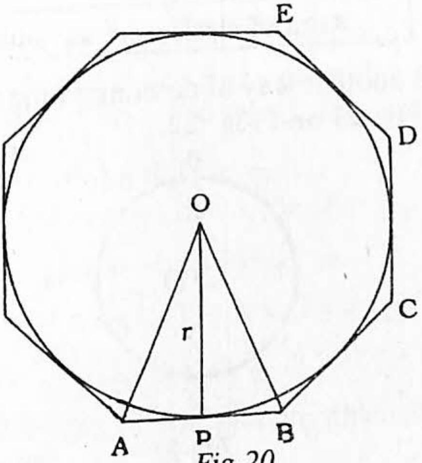


Fig-20

You can find the area of the polygon by the method of finding area of a triangle.

If radius of the circle be r units

Then the area of circumscribed polygon

$$= (\tfrac{1}{2}\,AB.r + \tfrac{1}{2}BC.r + \tfrac{1}{2}CD.r + \ldots)$$
$$= \tfrac{1}{2}\,(AB + BC + CD + \ldots.)\, r \text{ sq. units}$$
$$= \tfrac{1}{2} \text{ perimeter of the polygon } r \text{ sq units.}$$

Now increasing the number of sides of the polygon, we can make the perimeter of the polygon closer to the circumference of the circle and the area of a polygon having infinite number of sides will practically become equal area of the circle.

$\therefore$ Area of the polygon $= \frac{1}{2} \times$ perimeter of the polygon $\times\, r$

$= \frac{1}{2}$ circumference of the circle $\times\, r$

= Area of circle.

$\therefore$ Area of a circle $= \frac{1}{2}$ circumference of the circle $\times r$ sq. units

$$= \frac{1}{2} \times 2\pi r \times r = \pi r^2 \text{ sq units}$$

$$\boxed{\therefore \textbf{ Area of circle} = \pi r^2 \textbf{ sq units}}$$

There is another way of demonstrating this. Study the two figures Fig. 21 and Fig. 22.

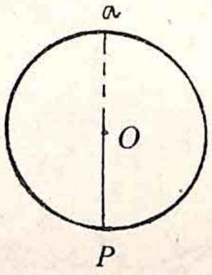


Fig-21

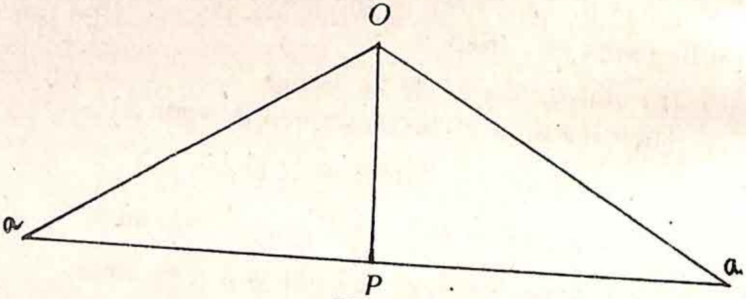


Fig-22

If the circle of fig. 21 is cut along Oa and spread out, it will take the shape of $\triangle aOa'$

Now if the radius of the circle is r units,
then OP = radius of the circle = r
and, aPa' = circumference of the circle = $2\pi r$ units

$\therefore$ In the $\triangle aOa'$, aPa' = base = $2\pi r$ units
and, OP = height = r units

Therefore, according to the formula of the area of a triangle

Area of $\triangle aOa' = \frac{1}{2} \times 2\pi r.\ r = \pi r^2$, = area of circle

Example 1. The diameter of a round table is 1·4 metres. What is the area of the upper surface of the table?

$$\text{Area of the surface} = \frac{22}{7} \times (0{\cdot}7)^2 \text{ sq.m.}$$
$$= 1{\cdot}54 \text{ sq.m.}$$

Example 2. A cow is tied to a post with a rope 3·5 m long. Find the area of the field that it can graze freely.

Area of the part of the field in which it can graze freely will be equal to the area of circle whose radius is 3·5 m.

$$\therefore \text{ Area of the plot} = \pi \times (3{\cdot}5)^2 \text{ sq. m.}$$
$$= \frac{22}{7} \times 3{\cdot}5 \times 3{\cdot}5 \text{ sq.m.}$$
$$= 38{\cdot}5 \text{ sq.m.}$$

Therefore, the area of the plot in which it can graze freely is 38·5 sq.m.

Area of an annular shaped region :

An annular region is bounded by circumferences of two circles. If the outer radius and the inner radius be R units and r units respectively then the area of the annular shased region will be $\pi\ (R^2-r^2)$ sq. units.

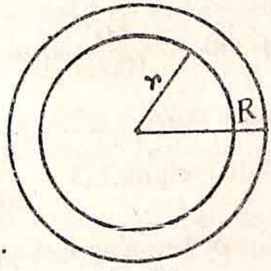


Fig-23

Area of an annular shaped region$=\pi(R^2-r^2)$ sq. units

Example 4. The diameter of a circular park is 56 metres. A circular path 3·5 m wide runs around outside the park. Find the cost of constructing the path at Rs. 8·40 per sq. metre.

Area of the path

=Area of the park including the path—Area of the park

$=\{\pi(31\cdot5)^2-\pi(28)^2\}$ sq.m.

$=\pi(31\cdot5+28)(31\cdot5-28)$ sq.m.

$=\frac{22}{7}\times59\cdot5\times3\cdot5$ (sq.m.)

$=654\cdot5$ sq.m.

∴ The total cost of constructing the path at Rs. 8·40 per sq.m. is Rs. (654·5×8·40)=Rs. 5497·80

Example 5. The internal and external diameters of an iron plate are 20 cm and 22 cm respectively. Find the area of the iron plate in it. What will be the weight of the ring if 1 sq.m. iron plate weighs 2 quintal 44 kg.?

The area of iron plate in the ring $=(\pi11^2-\pi10^2)$ sq. cm

$=\pi(11^2-10^2)$ sq. cm

$=\frac{22}{7}\times21$ sq. cm

$=66$ sq. cm.

You know, 66 sq. cm = $\frac{66}{10000}$ sq.m = $\frac{33}{5000}$ sq.m.

Now, 1 sq. m plate weighs 255 kg.

∴ $\frac{33}{5000}$ sq. m. plate weighs $225\times\frac{33}{5000}$ kg. = 1·683 kg.

∴ The weight of the ring is 1 kg. 683 gr.

Exercise 3

1. Find the area of the circles having the following radii : (i) 42 m (ii) 7 cm (iii) 6·3 m (iv) 4 m 9 dcm.

2. Find the area of the circle whose circumference is 44 cm.

3. What is the length of the fence of the circular garden whose diameter is 132 m? What is its area?

4. The area of a circular park is 616 sq. m. How much distance will have to be covered if you cross the park along its diameter?

5. A cow is allowed to graze with a rope, 3·5 m long tied to a pole. Had the rope been 4·2 m long, how much more area the cow could have grazed freely?

6. The fence of a half circular park is 144 m long. Find the area of the park.

7. There is a path around outside a circular playground. If the difference of the internal and the external circumferences of the path be 132 m, and its area be 9702 sq. m what will be the area of the playground?

8. An annular shaped iron plate has its internal and external diameters 18 cm and 32 cm respectively. Find the area of iron plate in it.

9. The area of a circle is 154 sq. cm. Find the area and the perimeter of the biggest square inscribed in it.

10. You have got a square shaped card-board sheet with 14 m sides. How much card-board will be waste, if you cut out 4 circular pieces of maximum size and of equal area from it?

11. Find the ratio of the perimeters of a circle and a square of equal area.

12. The area of an annular shaped brass plate is 269·5 sq. cm. If the external diameter of the ring be 28 cm, find its internal diameter.

13. A fort has a circular ditch around it. If the ditch be 7 m wide and its area be 29414 sq. m. find the area of the base the fort.

14. A triangular wooden plank equilateral in size has a surface area of 84 sq. cm. How much plank will be waste if a circular disc of 7 cm diameter is cut out of it?

15. A person finds that he takes 30 seconds less time if he walks along the diameter of the circular field instead of revolving once along its circumference. If his speed be 90 m per minute, calculate the area of the field.

16. An iron disc having 56 cm. diameter weighs 9 kg 856 gms. If 24 small discs of 7 cm diameter is cut out of it, how much plate will be waste? Calculate the weight of each of the small discs.

17. Out of a tin plate of uniform thickness three circular discs are cut-out. If the ratio of their diameters be 3 : 5 : 7, what will be the ratio of their weights?

MENSURATION

Miscellanious Exercise

1. Answer the questions attached to the following figures :

[Let area=A, length=l, breadth=b, height=h]

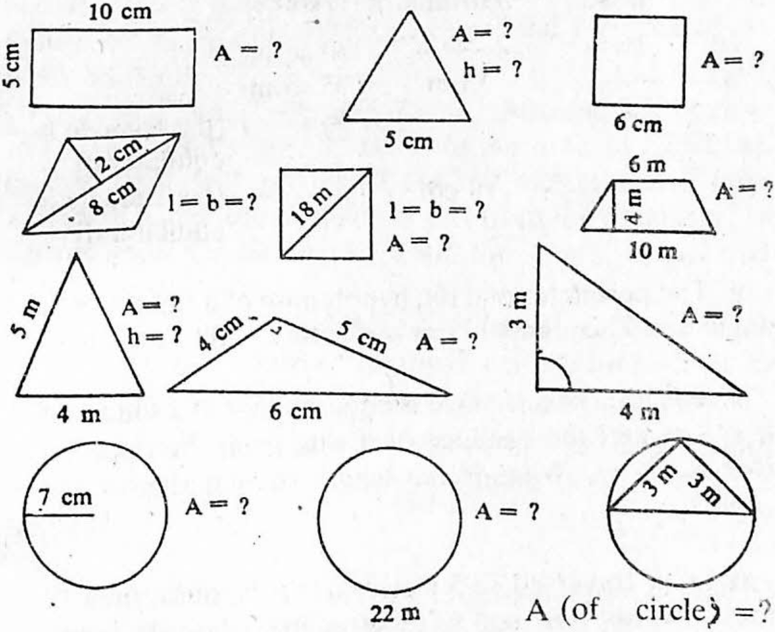


Fig-24

2. Find the perimeter of each of the following figures from the given data :

(i) three sides of a triangle are 11 cm, 13 cm and 15 cm.

(ii) Each of two equal sides of an isosceles triangle is 10 cm and the base is 7 cm.

(iii) In a right-angled triangle the sides containing the right angle are 21 dcm and 28 dcm.

(iv) The altitude of an equilateral triangle is $6\sqrt{3}$ metres.

(v) The area of an equilateral triangle is $4\sqrt{3}$ sq. metres.

(vi) Two adjacent sides of a parallelogram are 5 cm and 9 cm.

(*vii*) A diagonal of a square is 32 metres.
(*viii*) Two diagonals of a rhombus are 6 cm and 8 cm.
(*ix*) The area of a square is 42·25 sq. cm.
(*x*) The diameter of a circle is 14 mm.

3. Fill in the gaps : (in case of triangles)

	Base	Altitude	Area	
(*i*)	16 m	———	80 sq.m.	
(*ii*)	———	13 m	195 sq.m.	
(*iii*)	8 m	———	———	[the triangle is equilateral]
(*iv*)	———	18 cm	———	[the triangle is equilateral]

4. The perimeter and the hypotenuse of a right-angled triangle are 30 cm and 13 cm respectively, find the area.

5. The area of a triangle is equal to that of a square of side 4·5 m and the distance of a side from the opposite vertex is 7·5 m. What is the length of that side of the triangle?

6. Subir traversed 13·5 metres towards south, then 12 metres towards west and lastly 10 metres towards north. How far did he go direct from the first position?

7. In an isosceles triangle, the ratio of the base to one of the equal sides is 3 : 2 and the perimeter is 28 cm. Find the area of the triangle.

8. The lengths of three sides of a triangle are in the ratio 3:4:5 and the area is 7776 sq. metres. Find the lengths of three sides of the triangle and also the perimeter.

9. The perimeter of an equilateral triangle is 24 cm. What is the area of the triangle?

10. Three villages Gouripur, Lalpur and Jiarkandi (when taken to be points) are equidistant from each other.

To set a transformer, Makhan Babu is in search of a site which is equidistant from each of these three villages. If the distance of Jiarkandi from Gouripur is $2\sqrt{3}$ Km, find the position of that site.

11. The length and breadth of a rectangular garden are 85 metres and 65 metres respectively. A road, 1·5 metres wide, surrounds the garden. Find the area of the road.

12. B.T.P.S. Township Zone is surrounded by four roads AB, BC, CD, DA covering an area of 87696 sq. metres. $AB=442$ metres, $CD=370$ metres. Also it is known that AE, the distance of CD from the point $A=216$ metres and $EC=280$ metres. Find the lengths of AD and BC.

13. A square card-board is equal in area to a rectangular card-board of length 16 cm and breadth 4 cm. What is the length of each side of the square card-board?

14. How many tiles, each of length 40 cm and breadth 10 cm, are required to cover a square floor of side 16 metres?

15. Tulsi Babu purchased a rectangular plot of land 32 metres long and 25 metres broad. At the time of constructing the house he left off lands, 1·5 metres wide on three sides and also left off a portion of the land in front for gardening. He constructed the house on the remaining portion of the land. If the land on which the house is constructed is of square shape, what is its area? How much land is left off for gardening in front of the house?

16. The length of one side of a rhombus is 25 cm and that of a diagonal is 40 cm. What is the length of the other diagonal?

17. The length and breadth of a rectangular plot of land are in the ratio 3 : 2 and its area is 11094 sq. metres. Find the cost of fencing the plot of land at the rate of Rs. 2·50 per metre.

18. A rectangular plot of land is 150 metres long and 84·5 metres broad. The plot of land is divided into four plots by constructing two paths, each of 2·5 metres width, in the middle of the land, one lengthwise and the other breadthwise. What is the area of each plot? If for construction of the path one square metre costs Rs. 5·25, what is the total cost?

19. A plot of land in the shape of a trapezium is of area 1025 sq. metres. The lengths of two parallel ridges are 35 metres and 47 metres. If the plot of land is divided into two plots by a line that joins the vertices of two opposite angles, what will be area of each of the plots? Hence show that the ratio of two areas is equal to the ratio of two parallel sides.

20. What is the area of the largest circle that can be drawn in a square of side 21 cm?

21. The radius of a circular park is 210 metres. If Swagata wants to walk round the park at a speed of 5·5 km/hour, how much time will she take?

22. The circumference of a circular field is 88 metres. It is surrounded by a path, 3·5 metres wide. If the repairing cost of the path is Rs. 4·50 per metre, what is the total repairing cost?

23. The weight of an iron disc of diameter 35 cm is 4·5 kg. A right-shaped piece of width 3·5 cm is cut off from that disc. If the weight of the ring is $\frac{1}{3}$ [illegible] of the weight of the disc, find the internal and external diameters of the ring. As a result, the iron disc is divided into three pieces. What is the ratio of their weights?

CHAPTER II

Rectangular Parallelopiped

You have learnt how to find the perimeter and the area of plane figures like rectangles, squares etc. earlier in class VII. You have also learnt to find the perimeter and the areas of different triangles and circles in class IX. All of these are plane figures. But we have to use varieties of solid bodies in our daily life. They are of different shapes also. Again their upper surfaces are formed by one or more than one plane. A main characteristic of these bodies is that they occupy some space.

So to know the perimeters and areas of surfaces of the solid bodies is a necessity for us; so also the measurement of the space that they occupy. In this chapter we shall discuss the methods of measurements of these things.

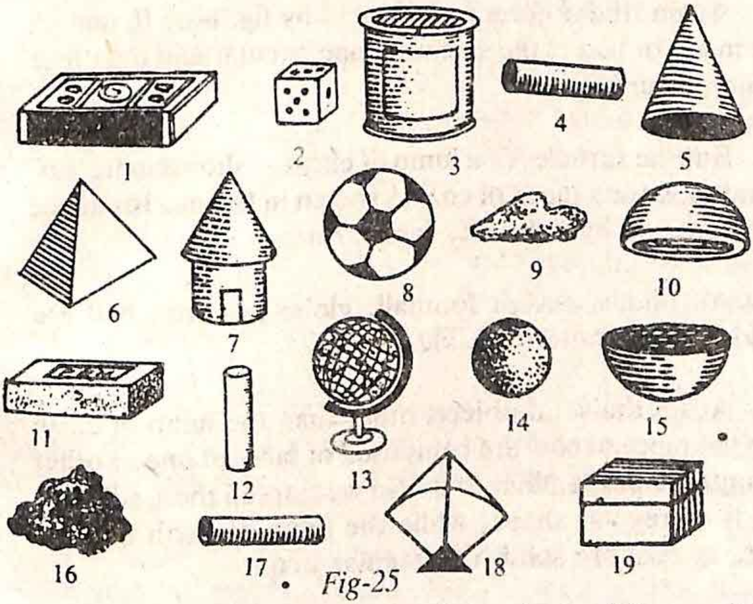


Fig-25

Have a look at the shapes and structures of these solid bodies. What do we see from the figures given?

Fig. No. 1 is the figure of a match box. You know a match box has 6 faces and each face is rectangular.

Similarly, the faces of a brick (fig. no. 11) and those of a water tank (fig. no. 19) are also rectangular. The ludo dice (fig. no. 2) has 6 faces, but they are of square shape.

Figure 3 represents a container. The two faces at the top and at the bottom are circular but it is surrounded by a curved surface. The shapes of the solid bodies (fig. nos. 4, 12, 17) are similar.

Figure 5 represents the porition of a *banana flower*. The face at the bottom is circular and the curved surface all around meets at a vertex. Similar is the shape of the upper part of a tent shown in fig. no. 7.

The football shown in figure no. 8 is composed of only one curved surface. The objects shown in fig. nos. 13 and 14 are likewise.

Again, the objects represented by fig. nos. 10 and 15 are made of two plane surfaces, one circular and the other a curved surface.

But the surfaces of a lump of earth as shown in fig. no. 9 and those of a piece of coal as shown in fig. no. 16 cannot be expressed by a definite familiar name.

All bodies except football, globe and iron ball are solids having more than one surface.

Again the solid objects other than the lump of earth and the piece of coal are composed of faces of one or other definite shapes familiar to us. So we can call these solids as solids of regular shape, while the lump of earth and the piece of coal are solids of irregular shape.

In this chapter we will discuss about the perimeter and area of faces of some regular solids. Besides that we will also discuss about their volume, that is, the measurement of the space they occupy.

Rectangular parallelopiped (or cuboid) :

Each face of the solids like fig. nos. 1, 2, 11 and 19 is rectangular and *any two* adjacent faces are mutually perpendicular. These solids are said to be *rectangular parallelopipeds or cuboids.*

Althouth the object in fig. 2 is of cuboidal shape, it has another characteristic. All its faces are squares. This is said to be a *cube.* The length, breadth and height of a cube are equal and we call each of them as a side of the cube.

Some parts of a rectangular parallelopiped and their description:

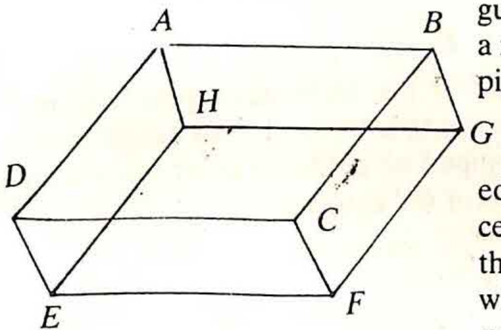


Fig-26

The adjoining figure *ABCDEFGH* of a rectangular parallelopiped has 8 *vertices.*

It has also 12 edges where two adjacent faces meet. Of these edges those, which are along length, are equal; those, which are along breadth, are equal; so also those which are along height.

It has also 6 rectangular faces. Any one of them is parallel to the opposite one and they are equal in area. In case of a cube, however, all the faces are equal in area.

The face on which the rectangular parallelopiped stands may be called its *base.* With respect to this base some edges are perpendicular to it. The length of each of these edges may be called the *height* of the rectangular parallelopiped. The line segment that joins one of the vertices and the vertex opposite to it is said to be a *diagonal* of the rectangular parallelopiped.

Think for a while and say how many diagonals you can draw in a rectangular prallelopiped.

Length of the diagonal of a rectangular parallelopiped:

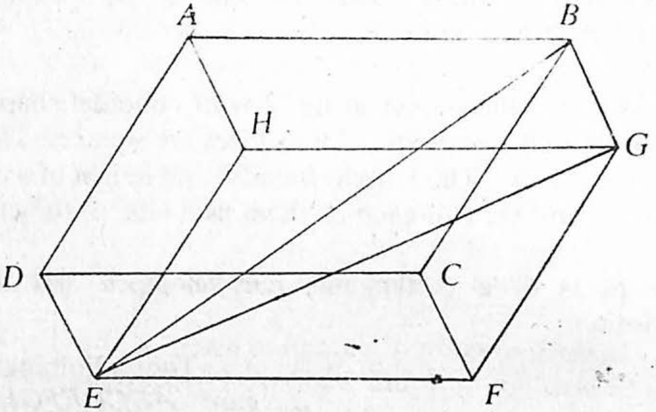


Fig-27

BE is a diagonal of the rectangular parallelopiped shown above. Let the length, breadth and height of the rectangular parallelopiped be a unis, b units and c units. Let us find the length of the diagonal BE.

E, G are joined.

Now, $\triangle BEG$ is a right-angled triangle.

$$\therefore BE^2=EG^2+BG^2$$

Again you know, since $\triangle EFG$ is a right-angled triangle, therefore $EG^2=EF^2+FG^2$

$$\therefore BE^2=EG^2+BG^2$$
$$=EF^2+FG^2+BG^2$$
$$\therefore BE=\sqrt{EF^2+FG^2+BG^2}$$

that is $BE=\sqrt{a^2+b^2+c^2}$ units,

Since, EF, FG and BG are the length, breadth and height of the rectangular peralleopiped respectively.

The length of the diagonal of a rectangular parallelopiped $=\sqrt{\text{length}^2+\text{breadth}^2+\text{height}^2}$

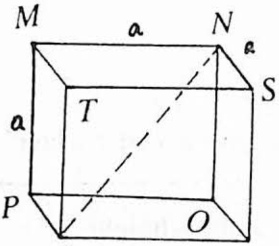


Fig-27(i)

NQ is a diagonal of the cube *MNOPQRST* as shown in the adjoining figure. Let the length of the side of the cube be *a* units. Then according to the above formula,

$$NQ=\sqrt{a^2+a^2+a^2} \text{ units}$$
$$=\sqrt{3a^2} \text{ units}$$
$$=a\sqrt{3} \text{ units.}$$

The length of the diagonal of a cube=side$\times\sqrt{3}$

Say, are the diagonals of a rectangular parallelopiped equal in length?

N.B.—Note that the length of the diagonal is the largest distance between any two vertices of a rectangular parallelopiped.

Total surface area of a rectangular parallelopiped:

The adjoining figure shows a rectangular parallelopiped whose length, breadth and height are *a*, *b* and *c* units respectively, *ABCD* and *EFGH* are the faces at the top and at the bottom respectively. Their total area= 2 × length × breadth$=2ab$ sq. units. Two lateral faces along the length are *DEFC* and

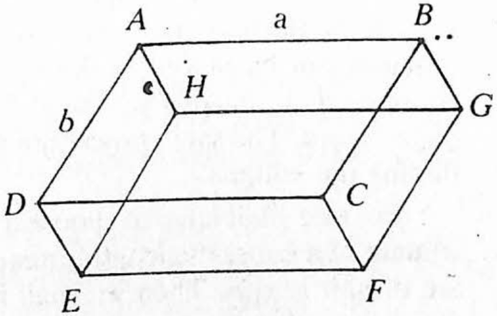


Fig-28

$ABGH$. Their area $= 2 \times$ length $\times$ height $= 2ac$ sq. units. Two lateral faces along the breadth are $ADEH$ and $BCFG$. Their total area $= 2 \times$ breadth $\times$ height $= 2bc$ sq. units.

$\therefore$ Area of the 6 faces of the rectangular parallelopiped
$=(2ab+2ac+2bc)$ sq. units
$=2(ab+ac+bc)$ sq. units

Total surface area of a rectangular parallelopiped
=2 (length×breadth+length×height+breadth×height)

In case of a cube, Length=breadth=height=side=a units

$\therefore$ Total area of the 6 faces of a cube
$=(2a^2+2a^2+2a^2)$ sq. units
$=6a^2$ sq. units

Total surface area of a cube = 6 × side^2

Remarks: Since length, breadth and height cannot be negative, we shall take the positive square root only to find the length of the side of a cube from the total surface area.

Volume of a rectangular parallelopiped:

You have seen, when we measured the area of a plane figure, we had first chosen a square region of definite value to be the unit and then we observed how many times this unit can be placed in the region of the plane figure considered. From this we could find out the area of the plane figure. The same procedure we will follow in case of finding the volume.

First we shall have to choose a cubic unit, which is the volume of a cube, the length, breadth and height of which are of unit length. Then we shall have to find how many times the cube chosen can be placed to cover the solid body, whose volume is to be found out. This number when associated with cubic unit will give the volume of the solid body.

Apply synthetic method and have a clear idea about the volume of a rectangular parallelopiped.

The length, breadth and height of the cube shown in fig. 29(i) are 1 cm each. When we measured the area we chose a square of length and breadth 1 cm each as unit and named that unit as 1 sq. cm. Similarly, we shall take the cube shown in Fig. 29(i) as unit and the volume of that cube is said to be 1 cubic centimetre.

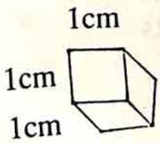


Fig-29

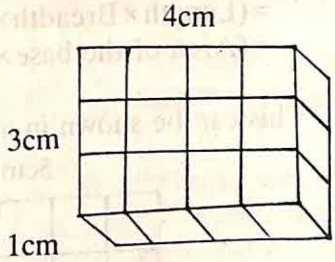


Fig-29(i)

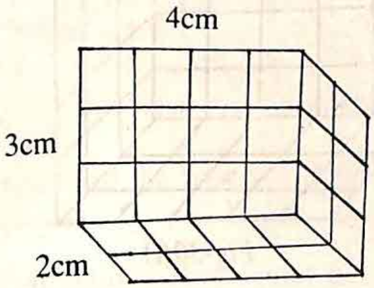


Fig-29(ii)

First we take 24 such cubes. Take 12 cubes from them and arrange them according to fig. no. 29(ii). The length, breadth and height of the figure formed are 4 cms, 3 cms and 1 cm respectively.

Now take the remaining 12 cubes. Place one each on each of the cubes as shown in fig. no. 29(iii).

Observe, you have obtained such a rectangular parallelopiped in fig. no. 29(iii), the length, breadth and height of which are 4 cms, 3 cms and 2 cms respectively. Its volume is obviously equal to that of 24 unit cubes.

Therefore, its volume is 24 cu. cm.

Again, observe that in the rectangular parallelopiped, Length × breadth × height = 4 cm × 3 cm × 2 cm = 24 cu. cm.

Hence we can say,
the volume of a rectangular parallelopiped
=(Length×Breadth×Height) cubic units
=(Area of the base×Height) cubic units

This can be shown in a different way:

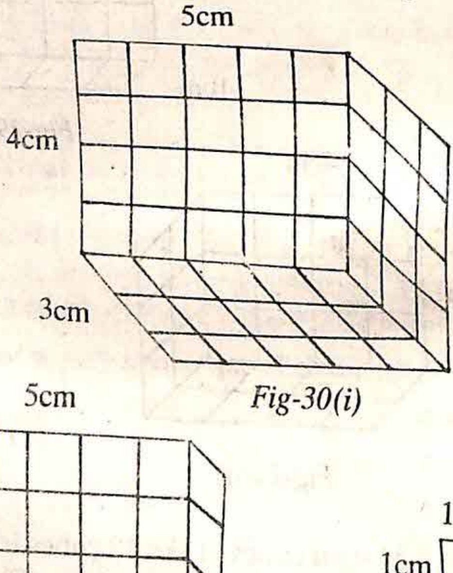


Fig-30(i)

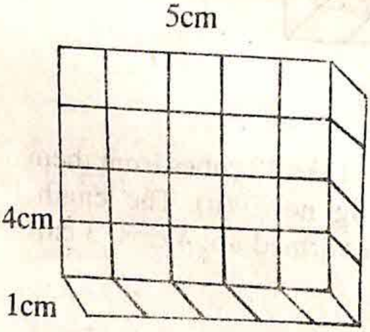


Fig-30(iii)

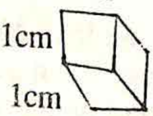


Fig-30(ii)

Fig-30 (i) shown above is of a rectangular parallelopiped. Its length, breadth and height are 5 cm, 4 cm and 3 cm respectively.

You have seen a sliced bread. Consider a bread of cuboidal shape. Cut it into thin slices of height 1 cm. You will get three such slices as shown in fig. 30(ii). Each slice will also be of cuboidal shape and its length, breadth and height are 5 cm, 4 cm and 1 cm respectively.

Now again if each slice is divided into 5 parts breadthwise and 4 parts lengthwise as shown in fig. no. 30(ii) we will get 20 pieces. Each piece will be a cube of length, breadth and height 1 cm each (see Fig. 30(iii).)

So we will get 60 such pieces from the three slices. Hence we can say the volume of this rectangular parallelopiped is 60 cu. cm.

See again, how the number 60 is associated with the measures of length, breadth and height of the rectangular parallelopiped. If we multiply 5 × 4 sq. units, which is the area of the base of the rectangular parallelopiped by 3 units, the height of the rectangular parallelopiped, we will get 20×3 cubic units i.e, 60 cu. units.

Therefore we can say, volume of a rectangular parallelopiped

=Area of the base × height

Volume of a rectangular parallelopiped
=Area of the base × Height

Since in case of a cube, Length=breadth=height=side

Volume of a cube=side^3

Example 1. The length, breadth and height of a rectangular parallelopiped are 8 cms, 5 cms and 3 cms respectively. Find its total surface area.

The total surface area of a rectangular parallelopiped

$=2(ab+bc+ca)$ sq. units

Where length=a units, breadth=b units and height=c units.

∴ Total surface area of the given rectangular parallelopiped

$=2(8\times5+5\times3+3\times8)$ sq. cm.

$=2\times79$ sq. cm.

$=158$ sq. cm.

Example 2. A cubical shaped water tank made of steel has its edge 2·2 metres, in length. The outer surfaces of the tank are painted at the rate of Rs. 25·50 per square metre. How much will be the total cost?

Total surface area of the cubical shaped tank$=6\times(2{\cdot}2)^2$sq.m.

$=29{\cdot}04$ sq. m.

∴ Total cost for painting the total surface area at the rate of Rs. 25·50 per sq. metre=Rs.(29·04 25·50) =Rs. 740·52.

Example 3 The length, breadth and height of a room are 5 metres, 4 metres and 3 metres respectively. What would be the maximum length of a piece of bamboo that can be placed diagonally in the room?

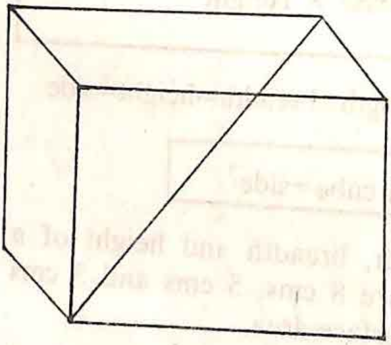


Fig-31

You understand that this is an example of a rectangular parallelopiped You also know the diagonals are the greatest distances between any two vertices of the rectangular parallelopiped . So the required length of the piece of bamboo is equal to the length of a diagonal.

Now, length of a diagonal $=\sqrt{a^2+b^2+c^2}$ units where length $=a$ units, breadth $=b$ units and height $=c$ units.

$\therefore$ The length of the diagonal of the room $=\sqrt{5^2+4^2+3^2}$ metres

$=\sqrt{50}$ metres

$=5\sqrt{2}$ metres.

$\therefore$ The maximum length of the piece of bamboo that can be placed diagonally in that room is $5\sqrt{2}$ metres.

Example 4. A tank, 5 metres long and 4 metres broad contains water, the depth of which is 2 metres. How many cubic metres of water does the tank contain?

The volume of water that the tank contains $=abc$ cubic units.

Where length $=a$ units, breadth $=b$ units, height $=c$ units.

$=(5\times4\times2)$ cubic metres

$=40$ cubic metres

Therefore, the tank contains 40 cubic metres of water.

Example 5. Two petrol tankers contained 18750 litres and 11250 litres of petrol respectively. Owing to formation of holes in the two tanks the petrol contained in them is poured in a large tank, of cuboidal shape to avoid leakage of oil. If the length and breadth of the large tank are 4 metres and 3 metres, what is the depth of the petrol poured in it?

Total quantity of petrol contained in two tanks = (18750 + 11250) litres=30000 litres.

Therefore the large tank will contain 30000 litres of petrol.

$\therefore$ 1 litre=1 cubic decimetre.

$\therefore$ 30000 litres of petrol=30000 cu. dcm of petrol.

Let the required depth of the petrol be h dcm.

$\therefore 40\times30\times h=30000$

$$\therefore h = \frac{30000}{40\times30} = 25.$$

That is, the depth of petrol = 25 dcm = 2·5 metres.

Example 6. The length and breadth of a rectangular land of paddy are 18 metres and 11 metres. That land is watered from an adjoined pond with the help of a pump. If the pump can sprinkle 39600 litres of water in an hour, how long will the pump work so that 3·5 decimetres of water will stand in the land?

If 3·5 decimetres of water stands in the land, then the volume of water will be (180×110×3·5) cubic decimetres and when expressed in litres the volume will be (180×110×3·5) litres. Again, since the pump sprinkles 39600 litres of water in an hour.

$\therefore$ the time for which the pump will work

$$= \frac{180 \times 110 \times 3{\cdot}5}{39600} \text{ hours}$$

$$= 1 \text{ hour } 45 \text{ minutes.}$$

Example 7. 40 planks, 2 m long, 2 dcm wide are sawed from a log of wood, 4 m long, 5 dcm wide and 3 dcm thick. 2% of wood is wasted on account of *cleaving*. But even then 108 cu. dcm of wood remain. Find the thickness of each plank.

The volume of wood that was in the long before cleaving was (40×5×3) cu. dcm=600 cu. dcm.

The volume of wood wasted on account of cleaving= $600 \times \frac{2}{100}$ cu. dcm=12 cu. dcm.

Let the thickness of each plank be x dcm. The volume of wood in 40 planks$=(20\times2\times x)\times40$ cu. dcm$=1600x$ cu. dcm.

Again the volume of wood, the log contains after cleaving = 108 cu. dcm.

$\therefore$ According to the conditions of the problem,

$$1600x=600-12-108$$

$$\text{or,}\quad 1600x=480$$

$$\text{or,}\quad x=\frac{480}{1600}=\cdot3$$

$\therefore$ The thickness of each plank$=\cdot3$ dcm$=3$ cm.

Example 8. A wooden box is made of plank, 4 cm thick. The box with its lid weighs 31 kg 920 gms. But it weighs 63 kg 168 gms. when it is filled with tea. The length, breadth and height of the box are 7 dcm, 5 dcm and 4·8 dcm respectively. Find the weight of a cu. dcm of wood and also that of tea.

The volume of the box$=(70\times50\times48)$ cu. cm
$=168000$ cu. cm.

The volume of the hollow space within the box
$=(70-8)\ (50-8)\ (48-8)$ cu. cm
$=104160$ cu. cm.

Therefore, the volume of wood by which the box is made
$=(168000-104160)$ cu. cm
$=63840$ cu. cm.

According to the conditions of the problem, 63840 cu. cm or 63·840 cu. dcm of wood weighs 31·920 kg.

$\therefore$ The weight of 1 cu. dcm of wood

$$=\frac{31{\cdot}920}{63{\cdot}840}\text{ kg}=0{\cdot}5\text{ kg}=500\text{ gms.}$$

Again, since the empty box weighs 31·920 kg. and the box filled with tea weighs 63·168 kg.

$$\therefore \text{ weight of tea only} = (63{\cdot}168-31{\cdot}920) \text{ kg}$$
$$= 31{\cdot}248 \text{ kg.}$$

Now, weight of 104160 cu. cm of tea, that is of 104·160 cu. dcm of tea is 31·248 kg.

$$\therefore \text{ Weight of 1 cu. dcm. of tea} = \frac{31{\cdot}248}{104{\cdot}160} \text{ kg}$$
$$= 0{\cdot}3 \text{ kg} = 300 \text{ gms.}$$

∵ The weight of 1 cu. dcm of wood is 500 gms and that of 1 cu. dcm of tea is 300 gms.

Exercise 4

1. The length, breadth and height of a harmonium box are 5·8 dcm, 3·2 dcm and 2·4 dcm respectively. What is the amount of expenditure to be incurred to polish the outer surface of the box all round at the rate of Rs. 1·25 per sq. dcm?

2. The length, breadth and height of the interior of a room are 5 m, 3·5 and 3 m respectively. The four walls and the roof are painted at the rate of Rs. 1·50 per sq. metre and the floor is polished at the rate of Rs. 15·50 per sq. metre. What will be the total cost?

3. The length, breadth and height of a steel trunk with lid are 5 dcm, 3·5 dcm and 2·4 dcm respectively. The total outer and inner surfaces are painted at the rate of Rs.0·80 per sq. dcm. What will be the total cost? (The thickness of the steel sheet is not to be taken into account.)

4. A wooden box is made of planks, 5 cm thick. The length, breadth and thickness of the box are 8 dcm, 6 dcm and 5 dcm respectively. The inner and outer sides of the faces of the box are painted at the rate of Re. 0·50 per sq. dcm. What will be the total cost?

5. A cuboidal shaped water tank has its length and height with the lid 4·5 metres and 1 metre respectively. The outer surfaces of the tank all round are painted at the rate of Rs. 5·75 per sq. metre and the amount of expenditure incurred is Rs. 241·50. What is the breadth of the tank?

6. The plastering of four walls and the roof of a room at the rate of Rs. 7·50 per sq. metre costs Rs. 671·55. If the length, breadth and height of the room are in the ratio 5:4:3, find its length, breadh and height.

7. The length, breadth and depth of a cuboidal shaped tank are 1 m 7 dcm, 1 m 5 dcm and 8 dcm respectively. How many cubic metres of water does the tank contain?

8. A cuboidal shaped pond is to be dug making its depth 4 metres. If the pond is 35 metres long; 25 metres broad, how many cubic metres of earth are to be removed? If the earth is removed at the rate of Rs. 4·25 per cubic metre, what will be the cost to dig the pond?

9. A petrol tanker contains 113960 litres of petrol. How many cuboidal shaped tubs will be required to keep that petrol, if the length, breadth and depth of each tub are 3·7 m, 2·8 m and 1 m respectively?

10. The water supply in a town is made from a cuboidal shaped reservoir, 25 m long, 17 m broad and 6 m deep. It is found in the morning that $\frac{8}{15}$th of the reservoir is filled with water. $\frac{1}{5}$th of the reservoir is found to be filled with water in the noon. How much water has been used during this period? (Assume that no more water was supplied during this period.)

11. A cuboidal shaped tank contains 10175 litres of water. The length and breadth of the tank are 3·7 m and 2·5 m respectively. What is the depth of water that is contained in the tank? (1 cu. dcm=1 litre)

12. A canal, 2 m wide and 8 dcm deep is dug by the Panchayet of a village to extend the scope of irrigation in the fields. It is found by calculation that 240 cubic metres of earth is removed. How long is the canal?

13. The length, breadth and height of the inner side of a box containing tea are 7·5 dcm, 6 dcm and 5·4 dcm respectively. When the box is filled with tea it weighs 52 kg. 350 gms., but when it is empty it weighs 3·75 kg. What is the weight of 1 cu. dcm of tea?

14. The length, breadth and thickness of a brick are 25 cm, 12·5 cm and 8 cm. How many cubic metres of space will be required to keep such 5500 bricks arranged? (It is to be assumed that there is no hollow space between two bricks.)

15. The length, breadth and height of a kerosine tin are 24 cm, 24 cm and 35 cm respectively. How many such tins can be filled with kerosine oil which is contained in a cuboidal tank, 3·15 metres long, 1·2 metres broad and 0·72 meter deep?

16. 30 cuboidal holes of equal depth are dug in the lands on both sides of the road that connects your village with the main road and the road is elevated by the earth removed from the holes. If the length and breadth of each hole are 14 m and 8 m respectively and if 2520 cubic metres of earth is required to elevate the road, what is the depth of each hole?

17. If 64 buckets of water are removed from a cubical shaped water tank, completely filled with water, $\frac{1}{3}$rd of the tank remains filled with water. If the length of each side of the tank is 1·2 metres, find how many litres of water each bucket holds, assuming that all buckets are of same measure.

18. A petrol tank of cuboidal shape contains such quantity of petrol that can fill 528 tins of 5 litres capacity. The depth of the tank is 7·5 dcm and breadth is $\frac{8}{11}$th of the length. Find the length and breadh of the tank.

19. The length, breadth and height of a packet of one gross match boxes are 2·8 dcm, 1·5 dcm and 0·9 dcm respectively. What is the volume of each match box? If the length of a match box is 5 cm and the breadth is 3·5 cm, what is its thickness? (1 gross=12 dozens)

20. Some bricks are arranged in a space of 25 cubic metres. If the length, breadth and height of each brick are 25 cm, 12·5 cm and 8 cm respectively, how many bricks are there in the pile of bricks? (Assume that there is no hollow space between two bricks.)

21. A cuboidal shaped water tank, 2·1 m long and 1·5 m broad is half filled with water. If 630 litres more water is poured into that tank, how much will the water level rise?

22. Three drums contain 800 litres, 725 litres and 575 litres of coconut oil. The oil contained in these three drums are poured into a tub of cuboidal shape. For this, the depth of oil in the tub becomes 7 dcm. If the length and breadth of the tub are in the ratio 4 : 3, find the length and breadth of the tub. If the depth of the tub would have been 5 dcm, then will it be possible to pour in that tub all the oil contained in a drum that contains 1620 litres of oil? Give reasons in support of your answer.

23. The daily water consumptions of the three families who are tenants of a three-storied building are 1200 litres, 1050 litres and 950 litres respectively. A reservoir is erected with such capacity that even after meeting the demand, 25% of water remains stored. For erecting the reservoir, a land, 2·5 m long and 1·6 m broad, only is available. How many metres deep will be the reservoir? If the breadth of the land would have been 4 dcm more, then what should have been the depth of the reservoir that would have served the purpose?

24. A tank, 3 m long, 2·5 m broad and 1·8 m deep is erected to soak bricks. Some bricks are soaked in it when the tank contains water 8 dcm deep. Bricks absorb $\frac{1}{5}$th of the quantity of water and even after that is is found that the water level has risen 5·2 dcm. If the length, breadth and thickness of each brick are 25 cm, 12·5 cm and 8 cm respectively, find the number of bricks that are soaked.

25. Corporation authorities have erected a water reservoir to supply pure water in a slum at Calcutta. The length, breadth and height of the reservoir are 4·3 m, 3·2 m and 1·5 m respectively. Everyday in the morning three tanks of same capacity are carried by a lorry and the reservoir is filled with

water contained in those three tanks. What is the capacity of each tank?

26. A low land, 48 m long and 31·5 m broad, is raised to 6·5 dcm. For this earth is removed from a hole, 27 m long and 18·2 m broad, dug by the side of the land. How deep will be the hole?

27. A wooden box is made of planks, 5 cm thick. The box with the lid weighs 115·5 kg. But when it is filled with rice it weighs 880·5 kg. The inner length and breadth of the box are 12 dcm and 8·5 dcm. One cubic decimetre of rice weighs 1·5 kg. What is the inner height of the box? What is the cost of painting outer side of the box all around at the rate of Re. 0·50 per sq. dcm?

28. The water level in a pond 20 m long and 18·5 broad is 3·2 m. if a pump can sprinkle 160 kilolitres of water per hour, in what time all the water of the pond can be sprinkled? If that water is thrown in a water-logged paddy field, how much will the depth of water in the field rise?

CHAPTER 4

RIGHT CIRCULAR CYLINDER

You have seen the figures 3, 4, 12 and 17 in page 33 Look at some more pictures of solid objects of similar shape.

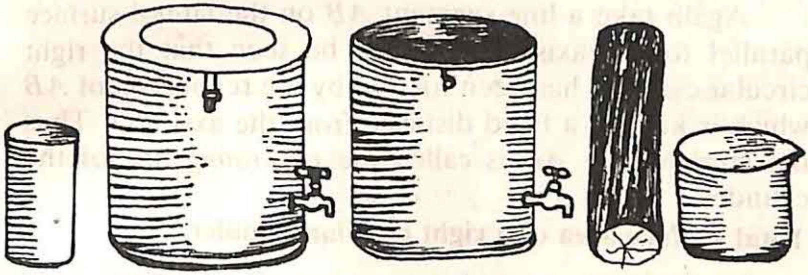


Fig-32

You can see that some of these objects are small and others, big. But the shapes of all are same. Solid objects of the above shape are examples of right circular cylinder.

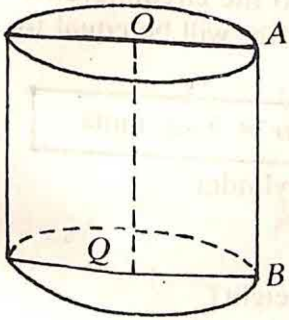


Fig-33

In the adjoining figure, a sketch of right circular cylinder is given. Observe some special features of the solid objects of the above shape.

First of all note that a right circular cylinder is formed by three faces. Of them, two faces—the upper and lower ones—are circular and congruent. The lower face on which the cylinder stands is said to be the *base* of the cylinder.

The third plane is a curved surface surrounding the cylinder. Further note that the curved surface stands perpendicularly along the circumference of the base. This face is called the *lateral surface* of the cylinder.

If the centres of two circular faces are O and Q respectively, the straight line OQ is called the *Axis* of the cylinder. There is a relationship between this axis and every point on the surrounding lateral surface. Every point on the lateral surface is equidistant from this axis.

The distance between two circular faces is called the height of the cylinder. Here OQ=height.

Again take a line segment AB on the lateral surface parallel to the axis OQ. It can be seen that the right circular cylinder has been formed by the revolution of AB which is kept at a fixed distance from the axis OQ. Thus the straight line AB is called the *generating line* of the cylinder.

Total surface area of a right circular cylinder:

Suppose, the radius of the circular faces of the above cylinder $=OA=QB=r$ and height $OQ=AB=h$.

Then, area of each of the two circular faces = πr^2 sq. units.

Now, if the curved lateral surface is cut along the generating line AB and stretched, a rectangle will be obtained whose length will be equal to the circumference of the circular face, i.e. $2\pi r$ and breadth will be equal to the height of the cylinder, h.

$\therefore$ area of the lateral surface$=2\pi r \times h$ sq. units

Area of the total surface of the cylinder

$= (2 \times \pi r^2 + 2\pi rh)$ sq. units

$= 2\pi r(r + h)$ sq. units

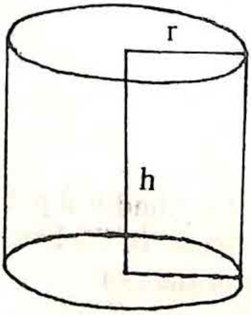
$= 2\pi \times$ radius (radius + height)

Total surface area of the cylinder

$=2\pi \times$ radius (radius + height)

$=2\pi r(r + h)$

Volume of the right circular cylinder:



Volume of the right circular cylinder

= Area of the base × height

$= \pi r^2 \times h$ cubic units

$= \pi r^2 h$ cu. units.

Volume of the cylinder = Area of the base×height
$= \pi r^2 h$

Example 1: Find the area of an iron-sheet necessary to make a cylindrical drum closed at both ends whose height is 6 dcm and radius of base is 3·5 dcm.

Total surface area
= 2 π × radius (radius+height),
$=\{2\pi \times 3{\cdot}5\,(3{\cdot}5+6)\}$ Sq. dcm.
$=\{2 \times \frac{22}{7} \times 3{\cdot}5 \times 9{\cdot}5\}$ Sq. dcm.
= 209 Sq. dcm.

∴ 209 Sq. dcm of iron-sheet will be required to make the drum.

Example 2: The radius of the base of a cylindrical tin-pot is 5·25 cm and the area of its lateral surface is 396 sq. cm. Find the height of the tin-pot.

Let the height of the tin-pot = h cm

Area of lateral surface
= Circumference of the base × height
$= (2\pi \times 5{\cdot}25 \times h)$ sq. cm.
$= 33\,h$ sq. cm.

But, according to the problem,

$$33h = 396$$

$$\therefore h = \frac{396}{33} = 12$$

$\therefore$ height of the tin-pot = 12 cm.

Example 3: The total surface-area of a cylindrical pot open at one end, is 1474 sq. cm. If the diameter of the base of the cylinder is 14 cm, find the height of the pot.

Let the height of the pot be h cm.

According to the problem, $\pi.7^2 + 2\pi.7h = 1474$

$$\therefore 154 + 44h = 1474$$

$$\therefore 44h = 1474 - 154$$

$$\therefore h = \frac{1320}{44} = 30.$$

$\therefore$ height of the pot = 30 cm.

Example 4: The diameter of the base of a drum 35 cm high is 24 cm. How much water does it contain?

Volume of the drum = Area of the base × height

$$= (\pi \times 12^2 \times 35) \text{ cu.cm}$$

$$= (\frac{22}{7} \times 12 \times 12 \times 35) \text{ cu. cm}$$

$$= 15840 \text{ cu. cm.}$$

$\therefore$ the drum can contain 15840 cu. cm of water.

Example 5: Find the number of tins, 4 dcm long, 2 dcm wide and 3 dcm high, which can be filled by the petrol contained in a petrol tanker, 8·4 m long and whose diameter of the base 1·2 m.

According to the condition of the problem,

$$(\pi \times {\cdot 6}^2 \times 8{\cdot}4) \div ({\cdot 4} \times {\cdot 2} \times {\cdot 3}) = ?$$

$\therefore$ required number of tins

$$= \frac{22}{7} \times \frac{{\cdot 6} \times {\cdot 6} \times 8{\cdot}4}{{\cdot 4} \times {\cdot 2} \times {\cdot 3}} = 396$$

∴ 396 tins can be filled up by the petrol of the tanker.

Example 6: A water-pipe of a pumpset, whose diameter is 14 cm, releases water at the rate of 2500 L.per minute. How much water will be irrigated by the pump in one hour. (1 litre=1 cu. dcm).

Cross-sectional area of the mouth of the pipe

$$= \frac{22}{7} \times 7^2 \text{ Sq. cm}$$
$$= 154 \text{ Sq. cm}$$
$$= 1{\cdot}54 \text{ Sq. dcm.}$$

Volume of the water irrigated by the pump in one minute

$$= (1{\cdot}54 \times 2500) \text{ cu. dcm}$$
$$= 3850 \text{ cu. dcm.}$$

Volume of water irrigated in 1 hour or 60 minutes

$$= (3850 \times 60) \text{ cu. dcm.} = 231000 \text{ cu. dcm.}$$

Now, 1 cu dcm. of water = 1 litre

∴ The pumpset can irrigate 231000 litres = 231 kilolitres of water in one hour.

Example 7: A water-pipe, whose diameter is 5·6 cm, releases water at the rate of 200 metres per minute. In what time will $\frac{5}{8}$th part of a cistern, 2·8 metres long, 2·2 metres broad and 1·6 metres deep, be filled by the water released by the pump?

Volume of the cistern = (28×22×16) cu. dcm. = 9856 cu. dcm.

∴ Volume of $\frac{5}{8}$ the part of the cistern = $9856 \times \frac{5}{8}$ cu. dcm. = 6160 cu. dcm.

Volume of water released in one minute

$$= \{\pi\,(2{\cdot}8)^2 \times 20000\} \text{ cu. cm}$$
$$= \left(\frac{22}{7} \times 2{\cdot}8 \times 2{\cdot}8 \times 20000\right) \text{cu. cm}$$

$= 492800$ cu. cm

$= 492{\cdot}8$ cu. dcm.

$\therefore$ Time required for filling $\frac{5}{8}$th part of the cistern

$= (6160 \div 492{\cdot}8)$ min

$= 12{\cdot}5$ min

$= 12$ min 30 Sec.

Example 8: On getting the news of fire in the village of Tilakpur, a fire brigade party came with a cylindrical water tanker. The diameter and length of the tanker are 2·8 metres and 6 metres respectively. The fire was extinguished in 40 min by three hose-pipes, the diameter of each being 2 cm, ejecting water at the rate of 420 metres per minute. How much water was used to extinguish fire? How much water is still left in the tanker?

Quantity of water in the tanker$=\{\pi\times(1{\cdot}4)^2\times6\}$ cu.m.

$= 36{\cdot}96$ cu.m $= 36960$ cu. dcm.

Water ejected by the pipe in one minute by one hose-pipe

$= (\pi \times 1^2 \times 42000)$ cu. cm.

$= 132000$ cu. cm.

$= 132$ cu. dcm.

$\therefore$ Quantity of water ejected by 3 hose-pipes in 40 min.

$= (132 \times 3 \times 40)$ cu. dcm

$= 15840$ cu. dcm.

$\therefore$ 15840 cu. dcm. of water have been used to extinguish fire. Quantity of water left in the tanker

$= (36960-15840)$ cu. dcm

$= 21120$ cu. dcm.

Example 9: 110 kilolitres of water has entered into the ship by a hole. After the hole is closed, a pump was taken for draining the water out of the ship. The diameter of the pipe of the pump is 10 cm and the velocity of water during the working condition is 350 metres per minute.

For how long should the pump work so that all water is thrown out of the ship?

In one minute, the pump can throw out,

$$=\left(\frac{22}{7} \times \frac{5}{10} \times \frac{5}{10} \times 3500\right) \text{cu. dcm of water}$$
$$= 2750 \text{ cu. dcm}$$

∵ Time required to throw out 110 kilolitres of water

$$= \frac{110000}{2750} \text{ min.} = 40 \text{ min.}$$

∴ The pump can drain out all water in 40 min.

Example 10: The inner and outer radii of a cylindrical pipe of iron, 6 metres long, are 3·5 cm and 4·2 cm respectively. What is the volume of iron in the pipe? If 1 cu. dcm of iron weighs 5 kilograms, what will be the weight of the pipe?

Outer radius of the pipe $= 4{\cdot}2 \text{ cm} \div 2 = 2{\cdot}1 \text{ cm}$
$= {\cdot}21 \text{ dcm.}$

Inner radius of the pipe$=3{\cdot}5 \text{ cm} \div 2 = {\cdot}175 \text{ dcm.}$

∴ cross-sectional area of the part of the pipe containing iron

$$= \pi\,({\cdot}21^2 - {\cdot}175^2) \text{ Sq. dcm.}$$
$$=\left(\frac{22}{7} \times {\cdot}013475 \text{ Sq.}\right)\text{dcm}$$

∴ Volume of iron contained in the pipe

$$=\left(\frac{22}{7} \times {\cdot}013475 \times 60 \text{ cu.}\right)\text{dcm}$$
$$= 2{\cdot}541 \text{ cu. dcm.}$$

∴ Weight of the pipe $= 2{\cdot}541 \times 5$ kg
$= 12{\cdot}705$ kg $= 12$ kg. 705 gm.

Exercise 5

1. A cylindrical waterdrum with lid has its diameter 6·3 dcm and height 9·1 dcm. What will be the cost for painting the outer surface of the drum at the rate of Re. ·60 per sq. dcm?

2. In the varandah of a room, there are three cylindrical pillars of diameter 4·2 dcm. The height of each pillar is 3·4 metres. What will be the cost of plastering the pillars at the rate of Rs. 15 per sq. metre?

3. The length of the cylindrical chimney of a workshop is 21 metres. The outer diameter of the chimney is 5·6 dcm. What will be the cost of painting the outer part at the rate of Re. 0·15 per sq. dcm.?

4. 93·06 sq. dcm of iron sheet is necessary to make a cylindrical drum, one mouth open. The diameter of the drum is 4·2 dcm. What is its height?

5. A cylindrical chimney, 28 metres long, is made of iron sheet, 56·32 sq. metres. What will be the diameter of its open mouth?

6. The outer diameter of a cylindrical iron pipe open at both ends is 8 cm and its length is 14·7 metres. What will be the cost for painting the curved surface of the pipe, at the rate of Re. 0/75 per sq. dcm?

7. The diameter and height of a beaker (a cylindrical glass used in laboratory) is 10·5 cm and 16 cm respectively. How many cu. cm of acid will the beaker contain?

8. The diameter of a drum for storing mustard oil is 8·4 dcm and its height 1·2 metres. How many cu. dcm of oil will five such drums contain?

9. The diameter and length of a milk tank of Mother Dairy is 1·75 metres and 6 metres respectively. How many litres of milk will the tank contain? (1 litre=1 cu. dcm.)

10. How much earth is to be dug out for constructing a well whose diameter is 2·1 metres and depth 8 metres?

11. How many cu. dcm of concrete materials will be necessary to construct two cylindrical pillars each of whose diameter is 5·6 dcm and height 2·5 metres? If the concrete materials are made of stone-chips, sand and cement in the ratio 4:2:1, how much of each of the components will be needed?

12. The diameter and height of a gas jar (cylindrical glass-containers for keeping gases in the laboratory) are 8 cm and 24·5 cm respectively. How many c.c. of gas will the gas jar contain?

13. The diameter and length of the cylindrical tank of corporation for supply of water in the slums are 1·4 metres and 7·2 metres respectively. How many drums each of 8·4 dcm. diameter and 1·25 m height, can be filled up by the water of the tank?

14. The diameter and length of a cylindrical wooden block of uniform density are 5·6 dcm and 2·5 metres respectively. What will be the weight of one cu. dcm of wood, if the total weight of the block is 9·24 quintals?

15. If a cylinder of fuel gas of diameter 2·8 dcm and length 7·5 dcm contains 15 kg 15 gms of gas, what will be the weight of 1 cu. dcm of gas?

16. The depth of water in a well, whose diameter is 1·8 metres is 5·25 m. How many litres of water are there in the well? (1 litre=1 cu. dcm).

17. 25 drums each of whose diameter is 1·4 metres and height 1·6 metres are filled with oil from a tank, full of oil, of diameter 2·1 metres and length 20 metres. How much oil will be left in the tank?

18. The heights of dilute acid in two beakers of equal diameters are 8 cm and 10 cm respectively. Acid from both the beakers is poured in a big beaker of diameter 10·5 cm and height of the acid becomes 8 cm. What are the diameters of the first two beakers?

19. The diameter and height of a kerosin tanker are 1·4 metres and 5 metres respectively. 250 tins each of length 24 cm, breadth 24 cm and height 35 cm are filled with oil from the tanker which was full with oil. How many lires will be left in the tanker? (1 cu. dcm=1 litre.)

20. Water is being drawn into a cistern, whose length, breadth and depth are 14·7 metres, 11 metres and 9 dcm respectively, by a pipe of diameter 4·2 cm at the rate of 250 metres per minute. In what time will the empty cistern be filled up with water?

21. A long gas jar of diameter 7 cm contains some water. An iron-cylinder of diameter 5·6 cm and length 5 cm is completely immersed in its water. How much will water-level be raised in the cylinder?

22. There are three cylindrical jars of equal diameters and equal heights. $\frac{2}{3}$rd part of the first jar, $\frac{5}{6}$th part of the second jar and $\frac{7}{9}$th part of the third jar are filled with dilute sulphuric acid. When the acid of these three jars are kept in another jar of diameter 2·1 dcm, the height of the acid becomes 4·1 dcm. If the diameter of the first three jars be 1·4 dcm, what is the height of each jar?

23. The depth of water in a brick-walled well, whose diameter is 2 metres, is 5·6 metres. A tank of length 4 metres, breadth 3 metres and depth 1·1 metres is filled by taking water from the well. What will be the depth of water-level in the well after drawing water? (Assume that no water enters into the well during this time.)

24. What part of a ditch, 48 metres long, 16·5 metres broad and 4 metres deep, can be filled by the stones and earth that will come out by digging a cylindrical tunnel of diameter 4 metres and length 56 metres?.

25. The pipe for the drainage of water from a roof, 13 metres long and 11 metres broad, was closed during rain fall. After rain, it was found that the depth of water-level on the roof is 7 cm. The out-let pipe has diameter 7 cm. Water is drawn out at the rate of 200 metres per minute from it. In what time will all water of the roof be drawn out by the pipe?

26. A pipe of 7 cm diameter is connected with a cylindrical tank of coal-tar which is 10·5 metres long. If coal-tar is drawn out at the rate of 210 metres per minute through the pipe, the tank full of coal-tar becomes empty in 45 minutes. What is the diameter of the tank?

27. The depth of a brick-walled well whose diameter is 2·8 metres, is 4 metres. At what rate per minute should all the water be pumped out from the well in 12·5 minutes, if the diameter of the pipe be 11·2 cm?

28. The cross-sectional area of a cuboidal wooden log, 2 metres long, is a square each of whose side is of length 14 dcm. If the log is to be converted into a cylindrical wooden block wasting minimum quantity of wood, how many cu. metres of wood will remain in it? How many cu. metres of wood will be wasted?

29. From a cylindrical woden stock, whose diameter and length are $12\sqrt{2}$ cm and 21 metres respectively, a cuboidal wooden log whose cross-sectional area is a square, is prepared. What quantity of wood will remain in the log? How much wood will be wasted?

30. The inner and outer diameters of a hollow cylindrical pipe of cast iron are 5·6 cm and 7 cm respectively. How many cubic dcm of iron will be contained in the above pipe if its length be 10 metres?

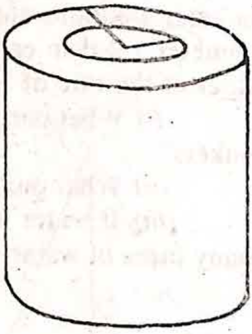


Fig-35

31. The inner and outer diameters of a cement-pipe required for constructing underground drainage system, are 1·05 metres and 1·4 metres respectively. How many cu. metres of concrete materials have been used for constructing such a pipe 200 metres long? If the ratio of quantities of cement, sand and stone-chips be 1:2:4, how many cu. metres of cement have been used in it?

32. The outer and inner diameters of a cylindrical gas-jar made of iron are 35 cm and 28 cm. How may cu. dcm of iron will be contained in the jar, if its height be 1·2 metres? What will be the weight of the gas-jar if one cu. dcm of iron weighs 5 kg?

33. It is required to make a plastering of sand and cement,

3·5 cm thick, surrounding four cylindrical pillars each of whose diameter is 17·5 cm. How many cu. dcm of plaster materials will be needed, if each pillar is 3 metres high? If the ratio of sand and cement in the plaster materials be 4:1, how many cu. dcm of cement will be needed?

34. The diameter of a cylindrical kerosine tanker is 2·8 metres. 50 drums, each of 8·4 dcm diameter and 1·5 meters height, are filled from the tanker full of kerosine. 500 tins, 24 cm long, 22 cm broad and 35 cm high can be filled by the kerosine left in the tanker. *(i)* What is the length of the tanker? *(ii)* How much oil was left after filling the drums? *(iii)* How many litres of oil are contained in each tin? (1 cu. dcm. = 1 litre).

35. $\frac{3}{5}$th part of a reserve tank, 14 metres long, 9 metres broad and 3·3 metres deep, is filled with drinking water. Water is supplied in different parts of the city by filling the cylindrical tankers, taking water from the reserve tank. A pump of diameter 1·2 dcm can fill up a tanker in 7 minutes by pouring water at the rate of 350 metres per minute.

(i) What part of the reserve tank was full, after filling 5 tankers?

(ii) What quantity of water does each tanker hold?

(iii) If water is drawn to fill up two more tankers, how many litres of water will remain in the reserve tank?

CHAPTER 5

RIGHT CIRCULAR CONE

You have seen the figure of a cone on page 6. Look at the pictures of some other objects similar in shape to that of a cone which are given below.

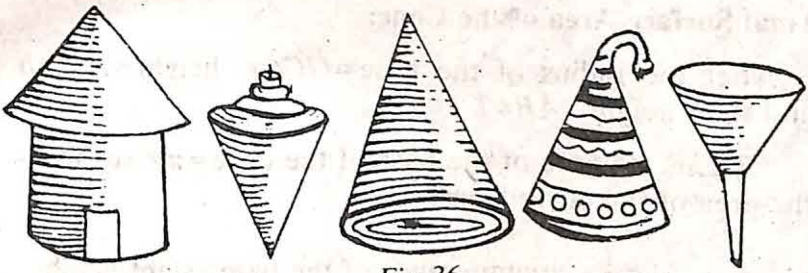


Fig-36

You observe that the shapes of figures bounded by thick lines are same. The solid figures of this shape are examples of right circular cones.

In the adjoining figure, the skeleton figure of a right circular cone is given. From the figure, you can understand that the face on which the cone stands is circular and the curved surface surrounding it has met at the vertex. The circular plane surface at the bottom is called the *base* and the curved surface surrounding it as the *lateral surface* of the cone.

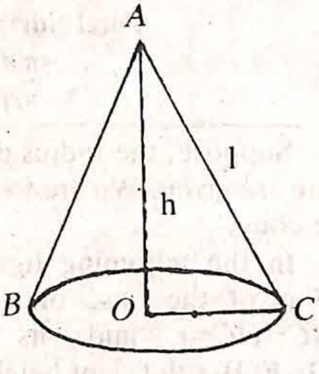


Fig-37

Further note from the above figure that the line segment joining vertex A to the centre of the base O is perpendicular to the base. This distance is called as the *height* of the cone. Here, AO=height.

Again the oblique distance AC or AB is called the *slant height* and OC the radius of the circular base.

Also note another point. If AB or AC is taken as a generating line, taking AO as axis, the cone is produced by revolving *AB around the axis AO.* AO and AB are, therefore called the *AXIS* and the *generating line* of the cone respectively.

Total Surface Area of the Cone:

Let the radius of the base$=OC=r$, height$=AO=h$ and slant height$=AB=l$.

Then, the area of the base of the cone$=\pi r^2$ sq. units the area of the curved surface

$$=\frac{1}{2}\times \text{circumference of the base} \times \text{slant height}$$
$$=\frac{1}{2}\times 2\pi r\times l \text{ sq. units}=\pi rl \text{ sq. units}$$
$\therefore$ area of the total surface of cone
$$=(\pi r^2+\pi rl) \text{ sq. units}=\pi r\,(r+l) \text{ sq. units}$$

Total Surface Area of Cone
$=\pi\times$ **radius (radius+slant height)**
$=\pi r(r+l)$

Suppose, the radius of the base and the height of the cone are given. We are to find out the total surface-area of the cone.

In the adjoining figure, the radius of the base of the cone $ABC=OC=r$ and its height $AO=h$. But the slant height is not given. We are to determine the total surface area of the cone. We are to know the slant height so that we can use the formula which we have just obtained.

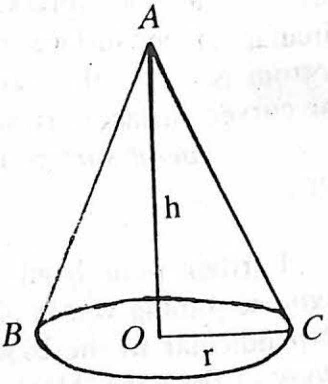


Fig-38

But here the magnitude of slant height AC is not given;

We can, however, get AC

from the right-angled triangle AOC.

Thus, $AC^2=AO^2+OC^2=h^2+r^2$

$\therefore AC=\sqrt{h^2+r^2}$

$\therefore$ area of the total surface of the cone

$=\pi r\ (r+\sqrt{h^2+r^2})$

Volume of Cone:

Volume of a cone$=\frac{1}{3}\times$ area of the base $\times$ height.

$=\frac{1}{3}\times\pi r^2\times h$ cu. units

$=\frac{1}{3}\ \pi r^2 h$ cu. units.

Volume of cone$=\frac{1}{3}$ area of the base$\times$height

$=\frac{1}{3}\ \pi r^2 h$

Example 1: The diameter of the base of a right circular cone is 7 cm and its slant height is 10 cm. Find the total surface area of the cone.

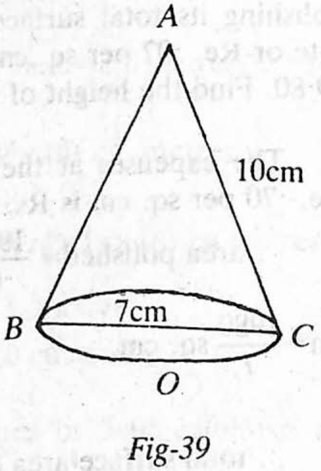


Fig-39

Diameter of the base of the cone $ABC=7$ cm

$\therefore$ radius$=r=3{\cdot}5$ cm
and slant height$=l=10$ cm.
$\therefore$ total surface area of the cone

$=\pi r(r+l)$

$=\pi\times3{\cdot}5(3{\cdot}5+10)$ sq cm.

$=\dfrac{22}{7}\times3{\cdot}5\times13{\cdot}5$ sq. cm.

$=148{\cdot}5$ sq. cm.

Example 2: 132 sq. m. of canvas is necessary for making the conical roof of a tent. The circumference of the base of the roof is 33 metres. What will be the slant height of the roof?

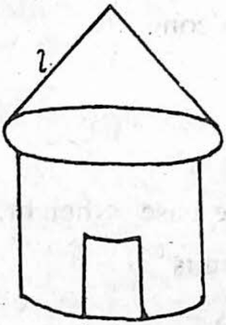


Fig-40

Suppose, the slant height of the roof=l metres. You know that, the area of the lateral surface of the cone=$\frac{1}{2}$ × circumference of the base × slant height.

$\therefore$ according to the problem

$$\tfrac{1}{2} \times 33 \times l = 132$$

$$\therefore\ l = 132 \times \frac{2}{33} = 8$$

$\therefore$ slant height of the roof=8 metres.

Example 3: The diameter of the base of a wooden toy having the shape of a right circular cone is 10 cm. The expenses for polishing its total surface at the rate or Re. ·07 per sq. cm is Rs. 19·80. Find the height of the toy.

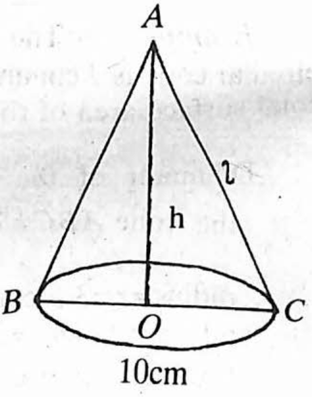


Fig-41

The expenses at the rate of Re. ·70 per sq. cm is Rs. 19·80.

$\therefore$ area polished= $\frac{19{\cdot}80}{{\cdot}07}$ sq. cm=$\frac{1980}{7}$ sq. cm.

$\therefore$ total surface·area of the toy=$\frac{1980}{7}$ sq. cm.

Let the slant height of the toy=l cm.

According to the condition of the problem

$$\pi.5(5+l)=\frac{1980}{7}$$

$$\text{or } \frac{22}{7}\times 5\times(5+l)=\frac{1980}{7}$$

$$\text{or, } 5+l=\frac{1980\times 7}{7\times 22\times 5}$$

or, $5+l=18$

or, $l=18-5=13$

$\therefore$ slant height of the toy=13 cm.

Let height of the toy=h cm.

$\therefore$ $h^2=l^2-5^2=13^2-5^2=169-25=144$

$\therefore$ $h=\sqrt{144}=12$

$\therefore$ height of the toy=12 cm.

Example 4: The diameter of the base of a conical water tank is 28 metres and the height is 18 metres. How much water does the tank hold? 1 litre = 1 cm. dcm.

Radius of the base of the tank=28 metres÷2=14 metres.

$\therefore$ Volume of the tank=$(\frac{1}{3}\pi\times 14^2\times 18)$ cu. metres.

$=(\frac{1}{3}\times\frac{22}{7}\times 14\times 14\times 18)$ cu.metres.

$=3696$ cu. metres.

$=3696000$ cu. dcm.

$\therefore$ the tank holds 3696000 litres or 3696 kilolitres of water.

Example 5: The radius of the base and the height of both the right circular cylinder and the right circular cone are 21 cm and 24 cm respectively. Compare their volumes.

Diameter of the base=21 cm.
∴ the radius of the base=10·5 cm.

Now volume of the right circular cylinder
$=$the area of the base×height
$=(\pi\times 10{\cdot}5^2\times 24)$ cu. cm.
$=(\frac{22}{7}\times 10{\cdot}5\times 10{\cdot}5\times 24)$ cu. cm.
$=8316$ cu. cm.

Again, volume of the right circular cone
$=(\frac{1}{3}\pi\times 10{\cdot}5^2\times 24)$ cu. cm.
$=2772$ cu. cm.

∴ Volume of cylinder : volume of cone $=8316 : 2732=3 : 1$.

In other words, having the same base and height, the volume of a cylinder is three times the volume of a cone.

[*N.B.* From the formulae, volume of the cylinder=area of the base×height, and, volume of the cone=$\frac{1}{3}$ × area of the base×height, the relation can also be obtained.]

Exercise 6

1. The diameter of the base of a right circular cone is 14 cm and the slant height is 12 cm. What is the area of the lateral surface of the cone?

2. The diameter of the base of a conical buoy is 3 metres and its height is 2 metres. What will be the cost of painting the total surface of the buoy, at the rate of Re. ·07 per sq. dcm?

3. $817\frac{1}{7}$ sq. cm. of tinfoil paper is required for covering the outer-surface of a conical sola hat. The diameter of the base of the sola hat is 20 cm. What is the height of the hat?

4. 77 sq. m. of tarpaulin is required to make a conical tent. The slant height of the tent is 7 metres. What will be the area of the base of the tent?

5. How much tarpaulin will be required to make a conical tent, 15 metres high, whose area of the base is $201\frac{1}{7}$ sq. metres?

6. The diameter of the base of a conical wooden toy is 1·4 dcm and its height is 1·2 dcm. How many cu. dcm of wood does the toy contain?

7. The inner diameter of the base of a conical tank is 21 metres and its slant height is 17·5 metres. How much water can the tank hold?

8. The area of the base of a conical tank is 13·86 sq. metres and its height is 2·8 metres. How many sq. metres of canvas has been used to prepare the tent? How many litres of air can the tent hold?

9. $75\frac{3}{7}$ sq. metres of iron-sheet is necessry to make a conical buoy. If the slant height of the buoy be 5 metres, how much air is present in the buoy? What is the height of the buoy? (neglect the breadth of the iron-sheet)

10. The diameter of the conical plumb (the instrument by which the mason finds the vertical line) of a mason is 3 cm. Its slant height is 2·5 cm. How many cu. cm of metal does the plumb contain?

11. The slant height of a conical mountain is 2·5 km. The area of the base of the mountain is 1·54 sq. km. What is the height of the mountain?

12. A right circular cylindrical drum contains some water. A conical iron piece of diameter 2·8 dcm and height 3 dcm is completely immersed in the water. The level of water in the drum is raised by 0·64 dcm. Find the diameter of the drum.

13. The diameters of a cylindrical drum and a conical iron piece is $2r$ dcm. The height of both the drum and the cone is h dcm. The conical iron piece is kept immersed in the drum

full of water and then removed. It is found that the height of water level in the drum is now $\frac{2h}{3}$. What conclusion can you draw from this? Explain with reasons.

14. The outer diameter of conical cap made of cork is 21 cm. At the rate of 2 paise per sq. cm, Rs. 11·55 is needed to cover the outer part of the cork cap with tin foil-paper. Find the height and the slant height of the cork cap.

15. The lower and upper part of the grannary (of paddy) of Anwar's family are cylindrical and conical respectively. The diameter of the base is 2·1 metres. The height of the cylindrical part is 2 metres and that of the conical part is 1·4 metres. How many cu. dcm of paddy will the grannary contain when it is one-third full with paddy?

CHAPTER 6

SPHERE

In Page 6, you have seen the pictures of a football and a globe. Look at some pictures below whose shapes are similar to those pictures.

shot put bal Globe Round Coconut Marble

Fig-42

Things of the above shapes are examples of sphere.

From the above pictures, you have certainly realised that a sphere is made by a single spherical curved surface.

The sketch of a sphere is given below.

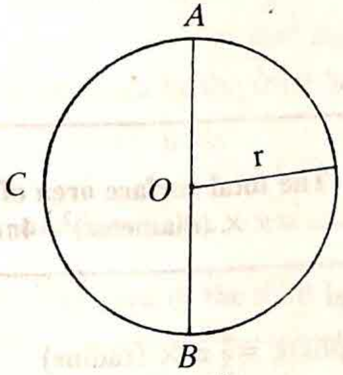


Fig-43

In the above figure, the point *O* lies in the middle of the globe such that its distance from any point on the sphere is equal. This point is the centre of the globe. The distance of any point on the sphere from the centre is known as its *radius*.

Look at the figure of the globe shown above. You can see a rod through the middle of the globe about which the globe can be rotated. Again, the straight line AB (Shown in the figure on the lower part that passes through the centre of the sphere and meets the curved surface in two opposite points is similar to that rod. The straight line AB is called the *axis* of the sphere.

A close look will show that the revolution of the semicircle ACB, taking the line AB fixed, has formed the sphere. Thus AB is the axis of the sphere here and the semicircle ACB is the *generating line.* Another characteristic feature of the sphere is that the intersection of any plane and the sphere will give a circular plane.

Total Surface Area of the Sphere:

If r is takne to be the radius of the sphere, its diameter will be $2r$. Now the area of the curve surface$=\pi\times(\text{diameter})^2$

$=4\pi r^2$ sq. units.

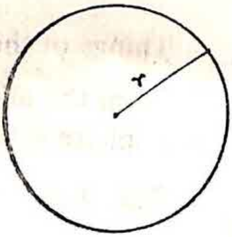


Fig-44

$\therefore$ **The total surface area of sphere $=\pi\times(\text{diameter})^2=4\pi r^2$**

Volume of Sphere:

Volume of sphere $=\frac{4}{3}\pi\times(\text{radius})^2$

$=\frac{4}{3}\pi r^3$ cu. units

$\therefore$ **The volume of sphere$=\frac{4}{3}\pi\times(\text{radius})^3=\frac{4}{3}\pi r^3$**

Hemisphere:

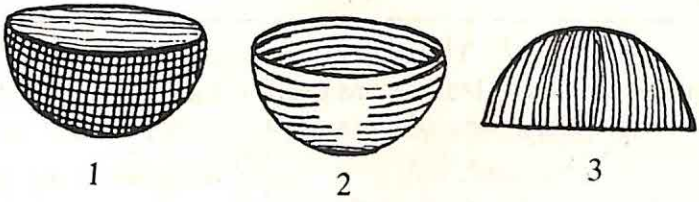


Fig-45

Among the pictures above, no. 1 is half of a metallic sphere, no. 2 is a hemispherical bowl and no. 3 is a dome. All of them are examples of hemispheres.

Solid hemispheres have two surfaces, one a circular surface and the second, a round curved surface.

Total Surface-area of a Solid Hemisphere:

Let the radius of the hemisphere be *r*.

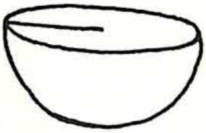


Fig-46

$\therefore$ Its area (of the circular plane) πr^2 sq. units and its surface area of curved part $= \frac{4\pi r^2}{2}$ sq. units

$= 2\pi r^2$ sq. units

So total surface-area of the solid hemisphere

$= (\pi r^2 + 2\pi r^2)$ sq. units

$= 3\pi r^2$ sq. units

$\therefore$ **Total surface area of the solid hemisphere $= 3\pi r^2$**

Volume of the Hemisphere:

The volume of the hemisphere will obviously be half of the volume of the sphere.

$\therefore$ The volume will be $(\frac{4}{3}\pi r^2 \div 2)$ cu. units

$= \frac{2}{3}\pi r^3$ cu. units

∴ Volume of the hemisphere $=\frac{2}{3}\pi r^3$

Example 1: The diameter of a football is 28 cm. How many sq. cm of leather are used to make the football?

Required area of leather $=(\pi \times 28^2)$ sq. cm

$=(\frac{22}{7} \times 28 \times 28)$ sq. cm

$= 2464$ sq. cm.

∴ 2464 sq. cm of leather are used to make the football.

Example 2: The diameter of the iron ball used for the shot-put game in your school is 14 cm. How many cu. cm of iron is there in the ball?

∴ diameter of the ball$=14$ cm. ∴ radius$=7$ cm.

∴ volume of iron present in the ball

$=(\frac{4}{3} \times \pi \times 7^3)$ cu. cm

$=(\frac{4}{3} \times \frac{22}{7} \times 7 \times 7 \times 7)$ cu. cm

$\frac{4312}{3}$ cu. cm

$= 1437\frac{1}{3}$ cu. cm

∴ $1437\frac{1}{3}$ cu. cm of iron is present in the ball.

Example 3: By melting three copper balls of radii 3 cm, 4 cm. and 5 cm., a big sphere is formed. What will be the radius of the big sphere?

Let radius of the big sphere be x cm.

∴ according to the condition of the problem,

$$\frac{4}{3}\pi x^3 = \frac{4}{3}\pi . 3^3 + \frac{4}{3}\pi . 4^3 + \frac{4}{3}\pi . 5^3$$

$$= \frac{4}{3}\pi . (3^3+4^3+5^3)$$

$= \frac{4}{3}\pi.(27+64+125)$

$\therefore x^3 = 27+64+125 = 216 = 6\times6\times6$

$\therefore x = 6.$

$\therefore$ radius of the big sphere=6 cm.

Example 4: The amount of water in a right circular cylinder of diameter 28 cm is such that three iron balls of equal diameter can be completely immersed in it. The level of water after the immersion of the balls is raised by 7 cm over the water level before their immersion. What is the diameter of each sphere?

Let radius of the spheres be r cm.

$\therefore$ according to the condition of the problem

$$3\times\frac{4}{3}\pi r^3 = \pi\times14^2\times7$$

$$\text{or, } 4r^3 = 14\times14\times7$$

$$\therefore r^3 = \frac{14\times14\times7}{4} = 7\times7\times7$$

$$\therefore r = 7$$

$\therefore$ radius of each sphere=7 cm

$\therefore$ their diameter=14 cm.

Example 5: A cylindrical drum of radius 21 cm and height 21 cm is full of water. What will happen if a iron of diameter 21 cm is completely immersed in it and then pulled out? What will be the ratio of the volumes of the drum and the sphere?

Radius of the drum=21 cm.

$\therefore$ volume of water in the drum

$= (\pi\times21^2\times21)$ cu. cm.

$= (\frac{22}{7}\times21\times21\times21)$ cu. cm

$= (22\times21\times21\times3)$ cu. cm

$= 29106$ cu cm.

When the sphere will be immersed in the water of the drum, the amount of water equal to the volume of the sphere will spout out of the drum.

Diameter of the sphere=21 cm.

∴ radius of sphere=10·5 cm.

∴ volume of sphere$=(\frac{4}{3}\,\pi\times 10{\cdot}5^3)$ cu.cm.

$$=(\frac{4}{3}\times\frac{22}{7}\times 10{\cdot}5\times 10{\cdot}5\times 10{\cdot}5) \text{ cu. cm.}$$

$$= 4851 \text{ cu. cm.}$$

When the sphere is completely immersed, the amount of water equal to the volume of sphere will spout out of the drum and the depth of water will decrease when the sphere is pulled out.

Let the new depth be h cm.

∴ according to the condition of the problem,

$$\pi\times 21^2\times h = 29106-4851$$

$$= 24255$$

$$\therefore\ \pi\times 21^2\times h=\frac{24255\times 7}{22\times 21\times 21}=17{\cdot}5$$

∴ depth of water will be 17·5 cm.

∴ volume of drum: volume of sphere=
29106 :4851=6:1

N.B.: Here the diameter of the sphere, the radius of the drum and the height of the drum are equal and each is $2r$ cm.

The volume of the drum$=4\,\pi r^2\,h$ cu.cm.
Since $h=2r$, the volume of the drum$=8\pi r^2$ cu. cm.

Again the volume of sphere$=\frac{4}{3}\,\pi r^3$ cu. cm.

∴ their ratio$=8\pi r^3 : \frac{4}{3}\pi\, r^3=8 : \frac{4}{3} =6:1$

Example 6: By flattening a solid brass sphere of radius 2·1 dcm., a circular sheet of 7 cm thickness is made. What is the diameter of the sheet? What will be the ratio of the total surface areas of the sheet and the sphere?

Let radius of the sheet be *r* cm.

∴ according to the condition of the problem.

$$\pi\times r^2\times 7=\frac{4}{3}\times\pi\times 21^3$$

$$\therefore\ r^2=\frac{\frac{4}{3}\times 21\times 21\times 21}{7}=4\times 21\times 21$$

$$\therefore\ r=\sqrt{4\times 21\times 21}=42$$

∴ radius of the sheet=42 cm.

∴ diameter of the sheet=84 cm=8·4 dcm.

Now, total surface area of the sheet

$$=2\pi\times 42\ (42+7)\text{ sq. cm}$$
$$=(2\times\frac{22}{7}\times 42\times 49)\text{ sq. cm}$$
$$=12936\text{ sq. cm.}$$

Total surface area of the sphere$=4\pi\cdot 21^2$ sq. cm

$$=\frac{22}{7}\times 21\times 21\text{ sq. cm}$$

$$=5544\text{ sq. cm.}$$

∴ Total surface area of the sheet : surface-area of sphere

$$=12936 : 5544 = 7 : 3$$

Example 7: The outer diameter of a hemispherical bowl made of brass of 1 cm thickness is 12 cm. If 1 cu. cm of brass weighs 8·85 gms, find the weight of the bowl.

volume of brass in the bowl

$=\frac{1}{2}\left(\frac{4}{3}\pi\times6^3-\frac{4}{3}\pi\times5^3\right)$ cu. cm

$=\left(\frac{2}{3}\times\frac{22}{7}\times91\right)$ cu. cm

$=\frac{572}{3}$ cu. cm.

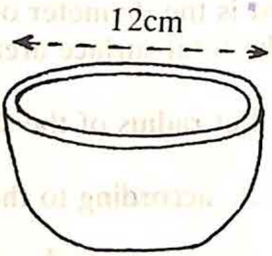


Fig-47

Since 1 cu. cm of brass weighs 8·85 gms,

Weight of the bowl $=\frac{572}{3}\times8{\cdot}85$ gms

$=1687{\cdot}4$ gms

$=1$ kg $687{\cdot}4$ gms.

Exercise 7

1. The diameter of a sphere made by an iron sheet is 14 cm. What will be the cost of painting it at the rate of Rs. 1·50 per sq. dcm?

2. 1 sq. dcm of leather costs Rs. 3·50. Rs. 86·24 is the cost of leather required for making a ball. What is the diameter of the ball?

3. For making a hemispherical bowl, 173·25 sq. cm. of metal sheet is necessary. Find the diameter of the outer rim of the bowl.

4. The radius of a solid iron ball is 2·1 cm. How many cu. cm of iron is there in the ball? What is the curved surface area of the ball?

5. The diameter of a solid lead sphere is 14 cm. How many spheres of 3·5 cm radius can be made by melting this sphere?

6. A brass block in the shape of a rectangular parallelopiped of length 6·6 dcm, breadth 4·2 dcm and thickness 1·4 dcm is melted. How many solid spheres of diameter 2·1 dcm can be made out of it? What is the volume of each ball in cu. dcm?.

7. The diameter of the base of a hemispherical dome is 42 dcm. What will be the cost of painting the outer surface of the dome, if the cost per sq. metre is Rs. 1·50?

8. The diameters of two hollow spheres made from the same metal sheet are 21 cm and 17·5 cm respectively. Find the ratio of the metal sheets required for making two spheres.

9. A cylindrical bar of diameter 2·8 cm is made from a solid sphere of gold of radius 4·2 cm. What is the length of the bar?

10. The radius of the cross-section of a solid cylindrical rod is 3·2 dcm. By melting the rod, 21 solid spheres are made. What will be the length of the rod if the radius of each of these spheres be 8 cm?

11. A cylindrical iron rod is made by hammering a solid iron ball of diameter 12 cm. If the length of the rod be 18 cm, what will be its diameter?

12. A cistern of length 21 dcm, breadth 11 dcm and depth 6 dcm is half filled with water. How many dcm will the water level be raised, if 100 iron balls each of diameter 21 cm be completely immersed in it?

13. The areas of curved surfaces of a right circular cylinder and a sphere are equal. If the radii of the cylinder and the sphere be equal, show that the ratio of their volumes is 3:2.

14. How many solid spheres each of 6 cm diameter can be made by melting a solid metallic rod of length 36 cm and diameter 4 cm?

15. A solid silver sphere of 6 dcm diameter is melted and circular sheet of thickness 1 dcm is prepared. What will be the diameter of the sheet?

TRIGONOMETRY

TRIGONOMETRY

What is Trigonometry and why ?

We know that "Necessity is the mother of invention" It is the compulsion of needs that is over-riding,. Thus it would not be unfair to look for such imperatives behind the birth of Trigonometry, which is a distinct branch of Mathematics. But what is this?

Whatever man finds in nature, he accepts after verification, be it from trees, fruits, flowers, animals that are within his reach to sun, moon, stars, planets of distant horizons, insurmountable mountains, vast oceans, rivers etc. Man has learnt once to measure things around him, has understood shapes of different objects with the help of geometrical figures, and has learnt to measure length, breadth and height. Also with the help of these, he has pursued ceaselessly his efforts to measure size, shape, distance, to determine laws of motion etc. of planets, stars, sun, moon etc. that are beyond his reach. It is because of such efforts that mathematicians could obtain the height of inacessible pyramid without getting at the top of it.

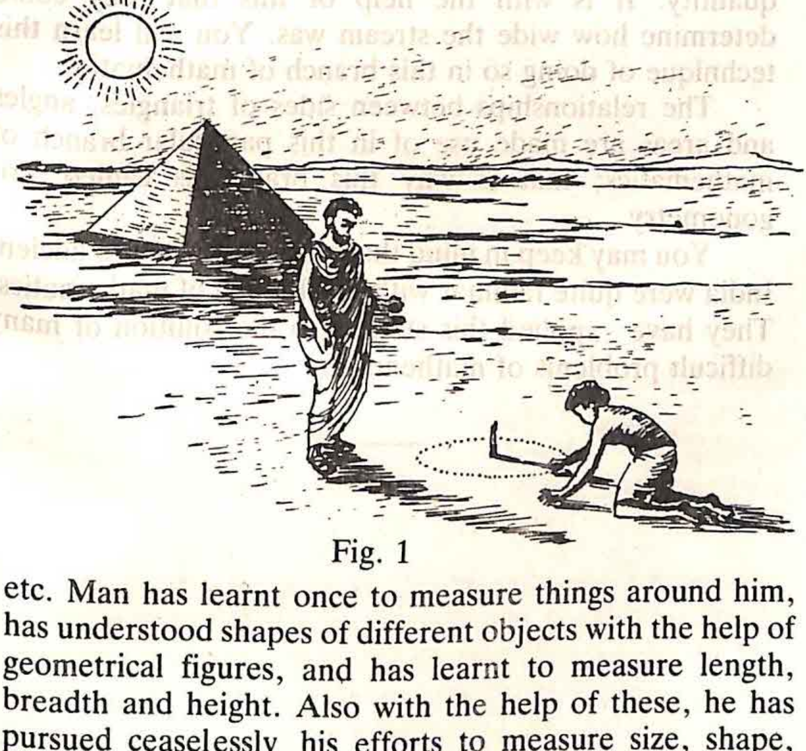


Fig. 1

Although man, in the course of development of civilisation has, on one hand, been confronted with various new problems, it is man again, on the other hand, who has discovered new methods of solutions; for example,

There is a hilly stream. One has to find out its width without using tape after crossing the stream. Man has done it with the help of mathematics.

In the meantime, mathematicians have been able to get at the relationships between sides and angles of triangles of various forms. They have come to know that the ratio of the sides of right-angled triangles, with respect to a definite measure of an acute angle, is a constant quantity. It is with the help of this that they could determine how wide the stream was. You will learn this technique of doing so in this branch of mathematics.

The relationships between sides of triangles, angles and areas are made use of in this particular branch of mathematics; that is why this branch is callled Trigonometry.

You may keep in mind that mathematicians in ancient India were quite familiar with this branch of mathematics. They have applied this subject to the solution of many difficult problems of mathematics.

CHAPTER I

MEASUREMENT OF ANGLE

As this particular branch of mathematics is mainly based on the ratios of sides of a right-angled triangle with respect to any of the two acute angles, we should have a full discussion about the angle.

What is an angle?

In Class VI you have learnt *(i)* an angle is formed at a point when two rays emerge from it. As in fig. 2 you see two rays OA and OB emerging from the point O form $\angle AOB$.

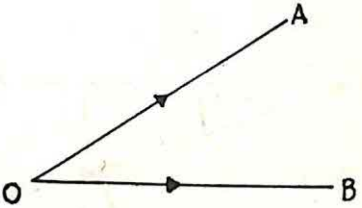


Fig. 2

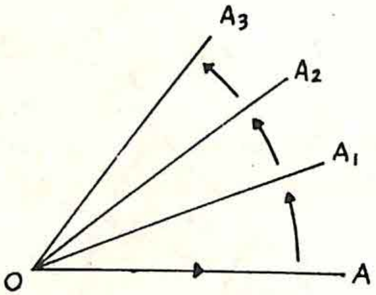


Fig. 3

(ii) If the initial point of a ray (the point from which the ray emerges) is kept fixed and the ray is rotated in the anticlockwise direction, then the subsequent positions of the ray make angles with the initial position at that fixed point.

In Fig. 3 the initial point O of the ray OA is kept fixed and the ray OA is rotated in the anti-clockwise direction to attain the positions OA_1, OA_2, OA_3 etc. Thus $\angle AOA_1$, $\angle AOA_2$, $\angle AOA_3$ etc. are formed at the point O.

Remark: We can similarly explain formation of $\angle A_1OA_2$, $\angle A_2OA_3$ etc.

Geometrical angle and Trigonometrical angle:

In trigonometry, we consider the magnitude of an angle and also the direction in which the rotating ray moves. Moreover, in trigonometry, the concept of angle is more comprehensive and full of variety than that in geometry.

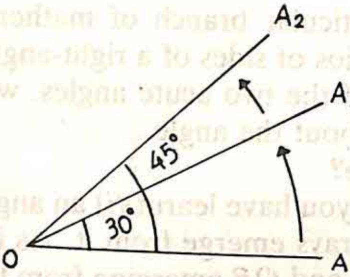
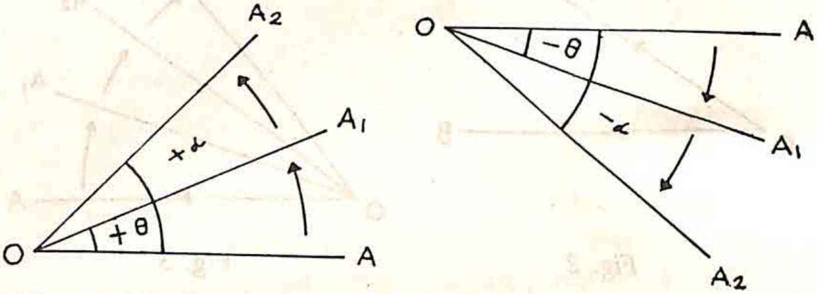


Fig. 4



(i) *Fig. 5* (ii)

(i) From the figures and the discussion above, it is clear that in geometry only the magnitude of the angle is the main thing that we consider. But, in trigonometry, we consider both the direction and the magnitude. The angles are positive when the ray rotates in the anti-clockwise direction and negative when it rotates in the clocckwise direction. As for instance, Fig. 5*(i)* shows that the ray has moved in the anti-clockwise direction and angles thus obtained are denoted by $+\theta$, $+\alpha$ etc.; but Fig. 5*(ii)* shows that the rotating ray has moved in clockwise direction to form angles $-\theta$, $-\alpha$ etc.

(ii) In geometry, the magnitude of an angle may be of

any value from 0° to 360°, and not more than that. That is, when the rotating ray makes one complete revolution it

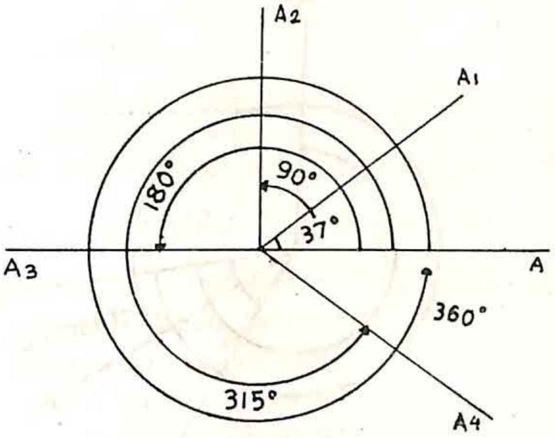


Fig. 6

forms an angle of 360°; but if it moves further, the magnitude of the angle will start from 0° and increase upto 360°.

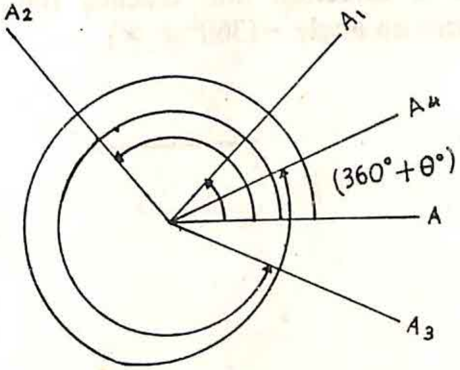


fig. 7

But in trigonometry, angles may be of any magnitude (positive or negative) starting from 0°. The magnitude of the angle increases by 360° for each complete revolution of the rotating ray. As for instance, in Fig. 7, when the rotating ray reaches the position OA_4, the magnitude of the angle so formed is $(360° + \theta°)$, but if the rotating ray traces an angle $\theta°$ $(0° \leqslant 0° \theta° \leqslant 360°)$ after two complete revolutions, the magnitude of the angle is $(2 \times 360° + \theta'°)$ $= (720° + \theta°)$.

Again, in Fig. 8 when the rotating ray rotates in the clockwise direction and reaches the position OA_3, it produces an angle $-(360^\circ + \propto)$.

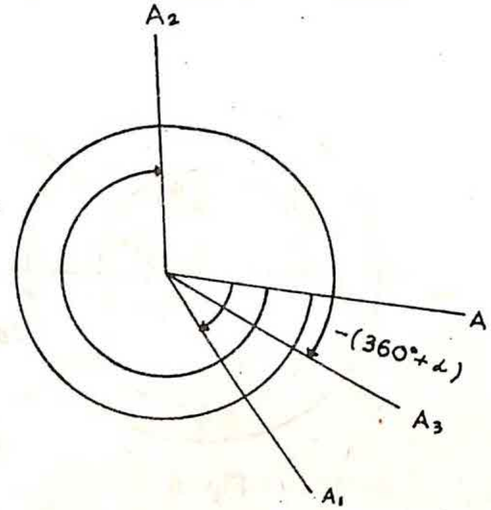


Fig. 8

Different systems of measurement of angles and their units :

Sexagesimal system :

In this system, the angle that the rotating ray traces making one complete revolution is taken to be 360°. One-fourth of it is 90°, that is one right angle = 90° and $\frac{1}{90}$ th part of a right angle is 1°. The other lower units of this system are minute and second. The following table shows the relation between them.

1 right angle = 90 degrees or 90°
1° = 60 minutes or $60'$
$1'$ = 60 seconds or $60''$

Circular system :

The unit of this system is based on the constant relation that exists between the circumference and the radius of any circle.

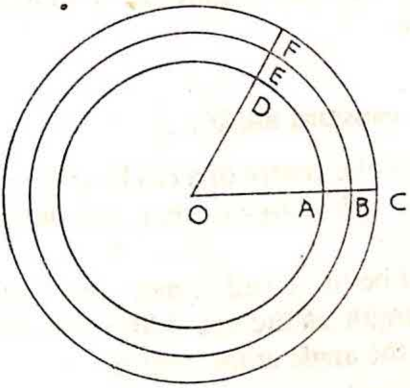


Fig. 9

Let us take three concentric circles. From the smallest circle let us cut off an arc AD equal in length to the radius of the circle. O, A and O, D are joined. Then $\angle AOD$ will be an angle at the centre subtended by the arc equal in length to the radius of the circle.

OA and OD are produced to meet other two circles at B, C and E, F respectively. On measurement you will find that BE and CF are equal in length to the radii of the corresponding circles. So $\angle BOE$ and $\angle COF$ are angles at the centre subtended by arcs equal in length to the respective radii.

Hence we may conclude that an arc of any circle equal in length to the radius of the circle subtends at its centre an angle of constant magnitude.

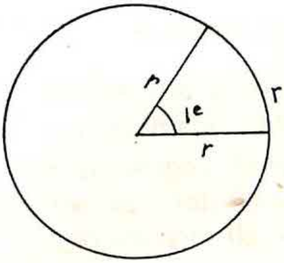


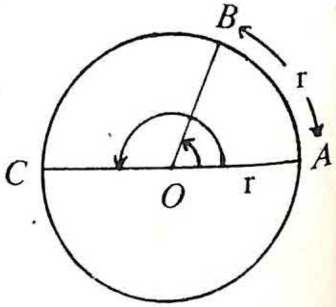
Fig. 10

This angle is taken as the unit of measurement of angles in the circular system. This is called one radian and written as 1^c. See the magnitude of an angle of one radian in the figure.

Radian is a constant angle :

Let O be the centre of a circle and radius $OR=r$. If we take an arc $AB=OA=r$, then by definition, $\angle AOB=1$ radian.

Let AO be produced to meet the circle at the point C. Then the length of the arc ABC=half the circumference and $\angle AOC$, the angle at the centre subtended by this arc=a straight angle = two right angles.



Now if we take the ration of the two arcs and that of the two angles, we have

$$\frac{\text{arc } AB}{\text{arc } ABC} = \frac{r}{\frac{1}{2}\times 2\pi r} = \frac{1}{\pi}$$

and

$$\frac{\angle AOB}{\angle AOC} = \frac{1 \text{ radian}}{2 \text{ right angles}}.$$

But in geometry, we can show that an arc of a circle is proportional to the angle it subtends at the centre of the circle.

Therefore, $\dfrac{\angle AOB}{\angle AOC} = \dfrac{\text{arc } AB}{\text{arc } ACB}$

Or, $\dfrac{1 \text{ radian}}{2 \text{ right angles}} = \dfrac{1}{\pi}$

$\therefore$ 1 radian $= \dfrac{2}{\pi}$ right angles.

This is constant as both 2 right angles and π are constants.

So, $\boxed{1 \text{ radian} = \frac{2}{\pi} \text{ right angles}}$

Corollary: π radian $= 2$ right angles
$= 180°$.

If we express one radian in the units of sexagesimal system, we will get

$$1 \text{ radian} = 57° \, 17' \, 44{\cdot}8'' \text{ (approx)}$$

Relation between the units of the two systems

Sexagesimal	Circular
360°	$= 2\pi$ radian $= 2\pi^c$
180°	$= \pi$ radian $= \pi^c$
90°	$= \frac{\pi}{2}$ radian $= \frac{\pi^c}{2}$

Remarks: (i) In the sexagesimal system, to indicate measures of angles in degrees symbol ' ° ' is used; for example, 60 degres = $60°$. And in the circular system, to indicate measures of angles in radians symbol ' c ' is used; for example, 1 radian = 1^c.

(ii) In trigonometry, we generally do not write the magnitude of angles like 1^c, but write them is terms of π.

When express in terms of π, we generally do not use the symbol ' c '.

Example 1: Two angles of a triangle are 75° and 45°. Find the value of the third angle in circular measure.

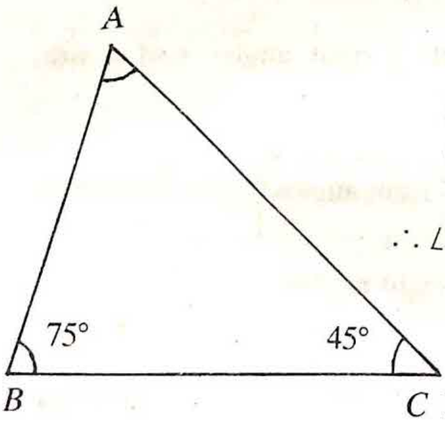


Fig. 12

In $\triangle ABC$, $\angle ABC = 75°$, and $\angle ACB = 45°$; $\angle BAC = ?$

You know that the sum of the three angles of a triangle is 180°.

$$\therefore \angle BAC = 180° - (75° + 45°)$$
$$= 180° - 120°$$
$$= 60°$$

Again, you know $180° = \pi$

$$\therefore 60° = \frac{60}{180}\pi = \frac{\pi}{3}$$

$\therefore$ *In* $\triangle ABC$, $\angle BAC = \frac{\pi}{3}$.

Example 2: A rotating ray revolves in the anti-clockwise direction and makes two complete revolutions and moves further to trace an angle of 30°. What are the sexagesimal and circular measures of the angle with reference to trigonometrical measure?

As the rotating ray moves in the anti-clockwise direction, the angle formed is positive.

You know, in one complete revolution the rotating ray traces an angle of 360°. So in two complete revolutions it makes an angle of 360° × 2 i.e., 720°. It has moved further to trace an angle of 30°. So the magnitude of the angle formed is (720°+30°) i.e., 750°.

$$\text{Now, } 180^\circ = \pi$$

$$\therefore\ 750^\circ = \frac{750}{180}\pi = \frac{25}{6}\pi = 4\frac{1}{6}\pi$$

Example 3: The ratio of the angles subtended at the centre by two unequal arcs of a circle is 5 : 3. If the magnitude of the second angle is 45°, find the sexagesimal and circular measures of the first angle.

Let the measure of the first angle be θ°

$$\text{Then, } \frac{\theta^\circ}{45^\circ} = \frac{5}{3} \qquad \therefore\ \theta^\circ = \frac{5}{3} \times 45^\circ = 75^\circ.$$

Again you know, $180^\circ = \pi$.

$$\therefore\ 75^\circ = \frac{75}{180}\pi = \frac{5}{12}\pi,$$

$\therefore$ the sexagesimal measure of the first angle is 75° and the circular measure is $\frac{5}{12}\pi$.

Example 4: *ABC* is an equilateral triangle in which *AD* is the line segment that joins the vertex *A* to the mid point of the side *BC*. What is the circular measure of $\angle BAD$?

As $\triangle ABC$ is equilateral,
$\angle BAC = 60°$

You also know that the median of an equilateral triangle bisects the corresponding vertical angle.

$\therefore \angle BAD = 30°$.

$\therefore$ the circular measure of $\angle BAD = \frac{30}{180}\pi = \frac{\pi}{6}$

Exercise 1

1. Express in degrees, minutes and seconds:

(i) $832'$ *(ii)* $7312''$ *(iii)* $375''$ *(iv)* $27\frac{1}{12}°$ *(v)* $72{\cdot}04°$.

2. Find the circular measures:

(i) $60°$ *(ii)* $135°$ *(iii)* $-150°$ *(iv)* $27°$ *(v)* $22°30'$ *(vi)* $-67°30'$ *(vii)* $52°52'30''$.

3. In a triangle one of the angles is $65°$ and the second one is $\frac{\pi}{12}$, find the sexagesimal and circular measures of the third angle.

4. The radius of a circle is 7 cm. Find the circular measure of the angle at the centre by an arc of this circle, 5 cm in length $\left[\text{Take } \pi = \frac{22}{7}\right]$.

5. The sum and difference of two angles are $135°$ and $\frac{\pi}{12}$, respectively. Find the sexagesimal and circular measures of the angles.

6. A rotating ray traces an angle $-5\frac{1}{12}\pi$. State in what direction the ray moves and find how many complete revolutions it makes and how much more in degrees the ray rotates.

7. The point A on a rotating ray OA moves to the position A_1, to make an angle $\frac{\pi}{6}$ at the point O. It moves further in the same direction to the position A_2 to make $\angle A_1OA_2 = 57°$. Find the sexagesimal and circular measures of $\angle AOA_2$.

8. A rotating ray OA revolves in the anti-clockwise direction and attains the position OA_1 to trace $\angle AOA_1 = 75°$ and then it moves in the clockwise direction to the position OA_2. The measure of $\angle AOA_2$ is found to be $\frac{\pi}{12}$. Find the sexagesimal measure of $\angle A_1OA_2$.

9. In an isosceles triangle ABC, $\angle ABC$ contained by two equal sides measures 45°. The bisector of $\angle ABC$ meets AC at the point D. Find the circular measures of $\angle ABD$, $\angle BAD$, $\angle CBD$ and $\angle BCD$.

10. $\angle ABC$ of a right-angled triangle ABC is 90° and $\angle BAC = \frac{3\pi}{8}$. *The perpendicular from the point* B on AC meets AC at the point D. Mention the names of all the angles of $\triangle ABD$ and $\triangle BCD$ and write down their circular measures.

11. Produce the base BC of an equilateral triangle ABC to the point E, such that $CE = BC$. Join A, E. Now mention the names of all the angles of $\triangle ACE$ and write down their circular measures.

CHAPTER 2

TRIGONOMETRICAL RATIOS

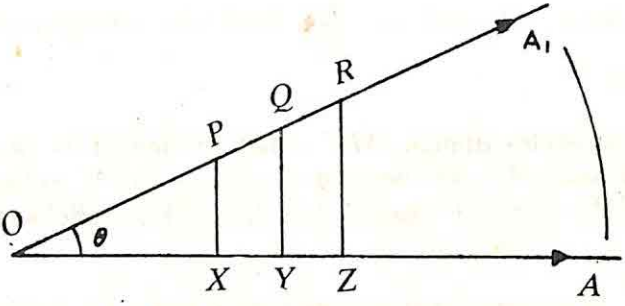


Fig. 13

Let the rotating ray OA attain a new position OA_1 revolving in the anti-clockwise direction to form $\angle AOA_1 = \theta$ and let it be an acute angle.

Now any numbr of points P, Q, R,....etc. are taken on OA_1 and perpendiculars PX, QY, RZ,.... etc. are drawn on OA from those points respectively. All the right-angled triangles POX, QOY, ROZ..... etc. are similar to each other.

Now from the properties of similar triangles we know,

(i) $\dfrac{PX}{OP} = \dfrac{QY}{OQ} = \dfrac{RZ}{OR} = ..$ *(ii)* $\dfrac{OX}{OP} = \dfrac{OY}{OQ} = \dfrac{OZ}{OR} = ..$

(iii) $\dfrac{PX}{OX} = \dfrac{QY}{OY} = \dfrac{RZ}{OZ} = ..$ *(iv)* $\dfrac{OP}{PX} = \dfrac{OQ}{QY} = \dfrac{OR}{RZ} = ..$

(v) $\dfrac{OP}{OX} = \dfrac{OQ}{OY} = \dfrac{OR}{OZ} = ..$ *(vi)* $\dfrac{OX}{PX} = \dfrac{OY}{QY} = \dfrac{OZ}{RZ} = ..$

Thus you see in a set of similar right-angled triangles with respect to same acute angle *(i)* the perpendiculars bear the same ratio to the corresponding hypotenuses; so also *(ii)* the bases to the corresponding hypotenuses, *(iii)* the perpendiculars to the corresponding bases etc. So we can say that the values of these ratios do not depend on the size of the triangles or the lengths of their sides. The values entirely depend on the magnitude of the acute angle θ. It is so becuse all the triangles are right-angled triangles having a common acute angle θ. Similar relations will hold whatever be the measure of the acute angle θ.

So we see that in similar right-angled triangles the ratios of any two sides, with reference to a common acute angle, give a definite value. This is the basis of trigonometrical ratios.

Again, we have seen, if we consider the ratio of any two sides of a right-angled triangle, we have six different ratios.

These six different ratios are identified by six names, one for each.

Trigonometrical ratios and their names:

Let, in $\triangle AOB$,

$\angle ABO = 1$ right angle,

$\therefore$ OA is the hypotenuse and with respect to the acute ange AOB, OB is the base and AB the perpendicular.

Let us suppose that $\angle AOB = \theta$.

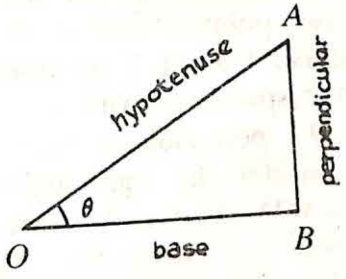


Fig. 14.

Then

$$\frac{AB}{OA} = \frac{\text{perpendicular}}{\text{hypotenuse}} = \text{Sine of } \theta \text{ or briefly } \sin\theta$$

$$\frac{OB}{OA}=\frac{\text{base}}{\text{hypotenuse}}=\text{Cosine of } \theta \text{ or briefly } \cos\theta$$

$$\frac{AB}{OB}=\frac{\text{perpendicular}}{\text{base}}=\text{Tangent of } \theta \text{ or briefly } \tan\theta$$

$$\frac{OA}{AB}=\frac{\text{hypotenuse}}{\text{perpendicular}}=\text{Cosecant of } \theta \text{ or briefly cosec } \theta$$

$$\frac{OA}{OB}=\frac{\text{hypotenuse}}{\text{base}}=\text{Secant of } \theta \text{ or briefly } \sec\theta$$

$$\frac{OB}{AB}=\frac{\text{base}}{\text{perpendicular}}=\text{Cotangent of } \theta \text{ or briefly } \cot\theta$$

N.B.: The side opposite to the angle under reference is to be taken as perpendicuular and the side adjacent to it except the hypotenuse as base.

Like all other ratios these ratios are also pure numbers and have no units.

In the beginning of this chapter you have become acquainted with the above property. Let us see it here more categorically.

Let us suppose that $\angle AOA_1 = \theta$ is formed due to change in position of the rotating ray OA to OA_1.

Here in Fig. 15, two points P and Q are taken on OA_1 and perpenddiculars PX and QY are dropped on OA from these two points respectively, while in Fig. 16, from two points R and S on OA, perpendiculars RM and SN are dropped on OA_1. Consider the right-angled triangles POX, QOY, ROM and SON. As one of the acute angles is θ, the other angle is $90° - \theta°$. So all these right-angled triangles are equiangular; that is, similar.

Now, according to the definitions of trigonometrical ratios,

$$\text{in } \triangle POX, \sin\theta = \frac{PX}{OP}$$

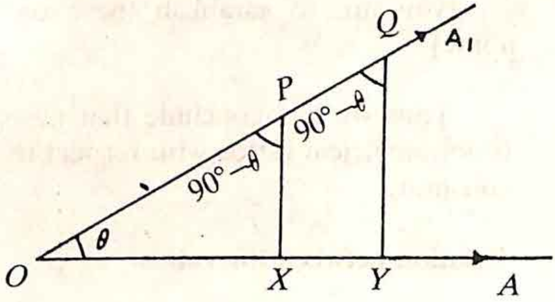


fig. 15

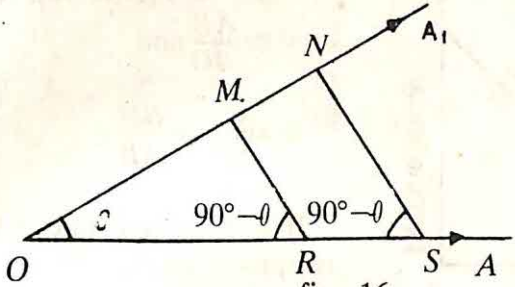


fig. 16

$$\text{in } \triangle QOY, \sin\theta = \frac{QY}{OQ}$$

$$\text{in } \triangle ROM, \sin\theta = \frac{RM}{OR}$$

$$\text{in } \triangle SON, \sin\theta = \frac{SN}{OS}$$

But, as the triangles are similar,

$$\therefore \frac{PX}{OP} = \frac{QY}{OQ} = \frac{RM}{OR} = \frac{SN}{OS}$$

So, we can say that the value of sin θ always remains the same and does not change for change in the sizes of the triangles or the lengths of their sides.

Similarly, this property can be established in case of cos θ , tan θ etc.

[you are to establish these on your own at your home].

Thus we can conclude that the value of each of the trigonometrical ratios with respect to a particular angle is constant.

Relation between the ratios:

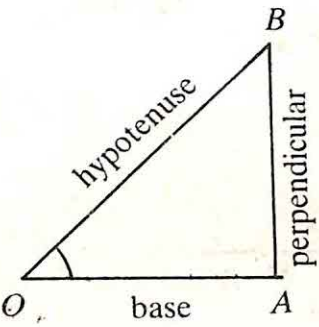


Fig. 17

From the figure, you see $\sin\theta = \frac{AB}{BO}$ and

$$\operatorname{cosec}\theta = \frac{BO}{AB}.$$

It is clear that one is the reciprocal of the other.

So, $\sin\theta = \frac{1}{\operatorname{cosec}\theta}$ and

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \qquad ..(i)$$

$$\text{Again, } \cos\theta = \frac{\text{base}}{\text{hypotenuse}} = \frac{OA}{OB} \text{ and}$$

$$\sec\theta = \frac{\text{hypotenuse}}{\text{base}} = \frac{OB}{OA}.$$

So, one is reciporcal of the other,

$$\text{that is, } \cos\theta = \frac{1}{\sec\theta} \text{ and } \sec\theta = \frac{1}{\cos\theta} \qquad ..(ii)$$

$$\text{Also, } \tan\theta = \frac{\text{perpendicular}}{\text{base}} = \frac{AB}{OA} \text{ and } \cot\theta = \frac{\text{base}}{\text{perpendicular}}$$

$= \frac{OA}{AB}$ which shows one is reciprocal of the other.

$$\therefore \tan\theta = \frac{1}{\cot\theta} \text{ and } \cot\theta = \frac{1}{\tan\theta} \qquad ..(iii)$$

Moreover, $\frac{\sin\theta}{\cos\theta} = \frac{\frac{AB}{OB}}{\frac{OA}{OB}} = \frac{AB}{OB} \times \frac{OB}{OA} = \frac{AB}{OA} = \tan\theta$

$$\therefore \frac{\sin\theta}{\cos\theta} = \tan\theta \qquad ..(iv)$$

and $\frac{\cos\theta}{\sin\theta} = \frac{\frac{OA}{OB}}{\frac{AB}{OB}} = \frac{OA}{OB} \times \frac{OB}{AB} = \frac{OA}{AB} = \cot\theta$

$$\therefore \frac{\cos\theta}{\sin\theta} = \cot\theta \qquad ..(v)$$

squares of trigonometrical ratios :

If we are to take square of sin θ, that is $(\sin\theta)^2$, we write it as $\sin^2\theta$. That is, we write $(\sin\theta)^2 = \sin^2\theta$. Similarly, $(\cos\theta)^2 = \cos^2\theta$, $(\tan\theta)^2 = \tan^2\theta$, etc.

Further discussions on relation between the ratios :

(1) From right-angled triangle AOB, in Fig. 17, we get

$$AB^2 + OA^2 = OB^2$$

Now, dividing both sides of the above relation by OB^2, we get

$$\frac{AB^2 + OA^2}{OB^2} = \frac{OB^2}{OB^2}$$

$$\text{or,} \quad \left(\frac{AB}{OB}\right)^2 + \left(\frac{OA}{OB}\right)^2 = 1.$$

But we know, $\frac{AB}{OB} = \sin\theta$ and $\frac{OA}{OB} = \cos\theta$

$$\therefore \sin^2\theta + \cos^2\theta = 1. \qquad .. (vi)$$

Corollary: $\sin^2\theta = 1 - \cos^2\theta$

$$\therefore \sin\theta = \sqrt{1 - \cos^2\theta} \qquad .. (1)$$

$$\cos^2\theta = 1-\sin^2\theta$$ Also

$$\therefore \cos\theta = \pm\sqrt{1-\sin^2\theta} \qquad \text{.. (2)}$$

(2) From the right-angled triangle AOB in Fig. 17, we get

$$OB^2 = AB^2 + OA^2.$$

Dividing both sides by OA^2, we get

$$\frac{OB^2}{OA^2} = \frac{AB^2}{OA^2} + \frac{OA^2}{OA^2} = \frac{AB^2}{OA^2} + 1$$

$$\therefore \quad \left(\frac{OB}{OA}\right)^2 = \left(\frac{AB}{OA}\right)^2 + 1.$$

But you know, $\frac{OB}{OA} = \sec\theta$ and $\frac{AB}{OA} = \tan\theta$

$$\therefore \sec^2\theta = \tan^2\theta + 1 \qquad \text{.. } (vii)$$

$$\therefore \sec\theta = \sqrt{\tan^2\theta + 1}$$

Corollary: $\tan^2\theta = \sec^2\theta - 1$

$$\therefore \tan\theta = \sqrt{\sec^2\theta - 1}.$$

(3) Again, dividing both sides of the relation $OB^2 = AB^2 + OA^2$ by AB^2, we get

$$\frac{OB^2}{AB^2} = \frac{AB^2}{AB^2} + \frac{OA^2}{AB^2} = 1 + \frac{OA^2}{AB^2}$$

$$\therefore \quad \left(\frac{OB}{AB}\right)^2 = 1 + \left(\frac{OA}{AB}\right)^2$$

Since, $\frac{OB}{AB} = \operatorname{cosec}\theta$ and $\frac{OA}{AB} = \cot\theta$

$$\therefore \operatorname{cosec}^2\theta = 1+\cot^2\theta = \cot^2\theta + 1 \quad .. \; (viii)$$

$$\therefore \operatorname{cosec}\theta = \sqrt{\cot^2\theta + 1}$$

Corollary: $\cot^2\theta = \operatorname{cosec}^2\theta - 1$

$$\therefore \cot\theta = \sqrt{\operatorname{cosec}^2\theta - 1}.$$

It is clear from the above discussion that if we know any one of the trigonometrical ratios for a particular angle, we can derive the values of all other ratios from the known one, for that particular angle. All these values are given in the table on page *29*.

N.B. You need not memorise these values. In doing a sum you are to find the values of other trigonometrical ratios from a given trigonometrical ratio by using a right-angled triangle. You may also use fundamental relations (i) to (viii).

Some solutions based on relations between the ratios :

Example 1: With respect to the positive acute angle θ in a right-angled triangle, $\sin\theta = \frac{12}{13}$, find the value of $\tan\theta$ and $\cos\theta$.

First method

In fig. 18, observe according to condition of the problem, $\triangle ACB$ be right-angled at B. Let $\angle ACB = \theta$ and with respect to it, we may assume perpendicular $AB = 12$ units and hypotenuse $AC = 13$ units, as

$$\sin\theta = \frac{12}{13}.$$

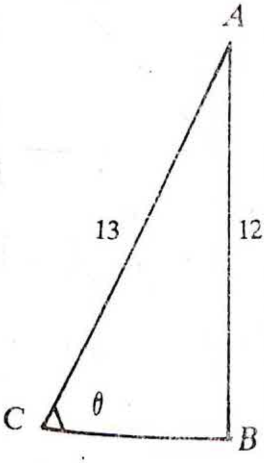


Fig. 18

$\therefore$ the base $BC = \sqrt{AC^2 - AB^2} = \sqrt{13^2 - 12^2} = \sqrt{25} = 5$

$$\therefore \tan\theta = \frac{\text{perpendicular}}{\text{base}} = \frac{AB}{BC} = \frac{12}{5}$$

$$\text{and } \cos\theta = \frac{\text{base}}{\text{hypotenuse}} = \frac{BC}{AC} = \frac{5}{13}$$

Second method :

$$\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \left(\frac{12}{13}\right)^2} = \sqrt{\frac{25}{169}} = \frac{5}{13}$$

$$\text{Again, } \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{12}{13}}{\frac{5}{13}} = \frac{12}{5}.$$

Example 2: If $\sin C = \frac{3}{5}$, $\cos C.\text{Cosec } C = ?$

First method :

In the right-angled triangle ABC, $\angle B$ is a right angle. Also let $AC = 5$ units, $AB = 3$ units

$$\therefore CB = \sqrt{AC^2 - AB^2} = \sqrt{5^2 - 3^2} \text{ units}$$
$$= 4 \text{ units}$$

$$\therefore \cos C.\text{cosec } C = \frac{4}{5} \times \frac{5}{3} = \frac{4}{3}.$$

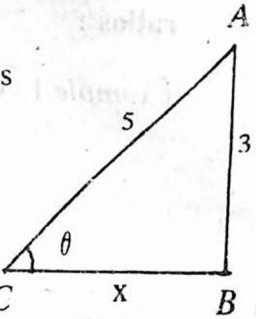


Fig-19

Second method :

$$\cos C.\text{cosec } C = \frac{\cos C}{\text{Sin } C} = \cot C = \sqrt{\text{cosec}^2 C - 1}$$

$$= \sqrt{\left(\frac{5}{3}\right)^2 - 1} \quad \left(\because \sin C = \frac{3}{5}\right)$$

$$= \sqrt{\frac{16}{9}}$$

$$= \frac{4}{3}$$

Example 3: In an isosceles triangle ABC, $AB = AC = 13$ cm and $BC = 10$ cm. Find sin $\angle ABC$, cos $\angle ABC$ and tan $\angle ABC$.

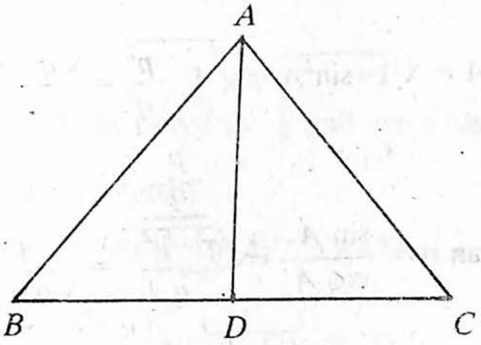


Fig-20

In fig. 20, in isosceles triangle $\triangle ABC$, $AB = AC = 13$ cm and $BC = 10$ cm.

Perpendicular AD is dropped from the verttex A on the base BC.

$\therefore BD = \frac{1}{2} BC = 5$ cm.

From the right-angled triangle ABD, we get

$$AD = \sqrt{AB^2 - AD^2} = \sqrt{13^2 - 5^2} \text{ cm} = 12 \text{ cm}$$

We know, $\angle ABC = \angle ABQ$

$$\therefore \sin \angle ABD = \frac{AD}{AB} = \frac{12}{13} = \sin \angle ABC$$

$$\cos \angle ABD = \frac{BD}{AB} = \frac{5}{13} = \text{Cos} \angle ABC$$

$$\tan \angle ABD = \frac{AD}{BD} = \frac{12}{5} = \tan \angle ABC$$

Example 4: If $\sin A = \frac{p}{q}$, find the values of $\tan A$, $\cot A$ and $\sec A$.

$$\cos A = \sqrt{1-\sin^2 A} = \sqrt{1-\frac{p^2}{q^2}} = \frac{\sqrt{q^2-p^2}}{q}$$

$$\therefore\ \tan A = \frac{\sin A}{\cos A} = \frac{\frac{p}{q}}{\frac{\sqrt{q^2-p^2}}{q}} = \frac{p}{\sqrt{q^2-p^2}}$$

$$\cot A = \frac{1}{\tan A} = \frac{\sqrt{q^2-p^2}}{p}$$

$$\sec A = \frac{1}{\cos A} = \frac{q}{\sqrt{q^2-p^2}}$$

Remark: You may also solve it by drawing a right-angled triangle. A being one of the acute angles and assuming perpendicular = p units and hypotenuse = q units.

Example 5: If $\cot \theta = \frac{x}{y}$, prove that

$$\frac{x\cos\theta - y\sin\theta}{x\cos\theta + y\sin\theta} = \frac{x^2-y^2}{x^2+y^2}$$

$$\text{L.H.S.} = \frac{x\cos\theta - y\sin\theta}{x\cos\theta + y\sin\theta}$$

$$= \frac{\frac{x\cos\theta}{\sin\theta} - y}{\frac{x\cos\theta}{\sin\theta} + y}$$

(Dividing numerator and denominator by $\sin\theta$)

$$= \frac{x\cot\theta - y}{x\cot\theta + y}$$

$$= \frac{x.\frac{x}{y} - y}{x.\frac{x}{y} + y} = \frac{x^2-y^2}{x^2+y^2} = \text{R H S.}$$

Example 6: If $\sin\theta+\cos\theta = \frac{7}{5}$ and $\sin\theta\cos\theta = \frac{12}{25}$ find the value of $\sin\theta-\cos\theta$.

$$(\sin\theta-\cos\theta)^2 = (\sin\theta+\cos\theta)^2 - 4\sin\theta\cos\theta$$

$$= \left(\frac{7}{5}\right)^2 - 4.\frac{12}{25} = \frac{49}{25} - \frac{48}{25} = \frac{1}{25}$$

$$\therefore \sin\theta-\cos\theta = \pm\frac{1}{5}$$

Example 7: Express $1+2\sin\theta\cos\theta$ as a perfect square.

$$1+2\sin\theta\cos\theta = \sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta$$
$$= (\sin\theta+\cos\theta)^2$$

Example 8: If $\frac{\sec\theta+\tan\theta}{\sec\theta-\tan\theta} = 2\frac{51}{79}$, find the value of $\sin\theta$.

$$\frac{\sec\theta+\tan\theta}{\sec\theta-\tan\theta} = 2\frac{51}{79}$$

or $\frac{\sec\theta+\tan\theta}{\sec\theta-\tan\theta} = \frac{209}{79}$

or $\frac{2\tan\theta}{2\sec\theta} = \frac{130}{288}$ (by componendo and dividendo)

or $\frac{\sin\theta}{\cos\theta} \times \cos\theta = \frac{65}{144}$

Example 9: $\triangle ABC$ is right-angled at B. The length of the hypotenuse of that triangle is $\sqrt{13}$ and the sum of the lengths of other two sides is 5. Find the value of $\sin C + \sin A$.

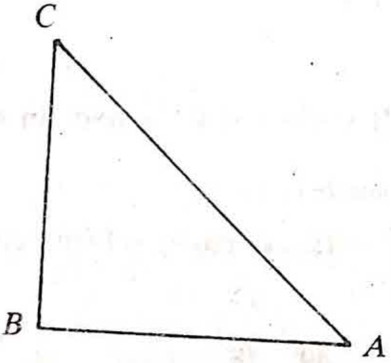


Fig-21

The hypotenuse of the right-angled triangle $ABC = AC$. AB and BC are perpendiculars with respect to $\angle C$ and $\angle A$ respectively.

$$\therefore \sin C + \sin A = \frac{AB}{AC} + \frac{BC}{AC} = \frac{AB+BC}{AC} = \frac{5}{\sqrt{13}}$$

Alternative method :

Let $AB = x$, then $BC = 5 - x$

$\therefore x^2 + (5-x)^2 = (\sqrt{13})^2$

or, $x^2 + 25 - 10x + x^2 = 13$

or, $2x^2 - 10x + 12 = 0$

or, $x^2 - 5x + 6 = 0$

or, $(x-3)(x-2) = 0 \therefore x = 3$ *or* 2

$\therefore$ when $AB = 3$, $BC = 5 - 3 = 2$

$$\therefore \sin C = \frac{AB}{AC} = \frac{3}{\sqrt{13}}, \sin A = \frac{BC}{AC} = \frac{2}{\sqrt{13}}$$

Inter-relations between the trigonometrical ratios

	sin A	**cos A**	**tan A**	**cot A**	**sec A**	**cosec A**
$\sin A =$	$\sin A$	$\sqrt{1-\cos^2 A}$	$\dfrac{\tan A}{\sqrt{1+\tan^2 A}}$	$\dfrac{1}{\sqrt{1+\cot^2 A}}$	$\dfrac{\sqrt{\sec^2 A-1}}{\sec A}$	$\dfrac{1}{\operatorname{cosec} A}$
$\cos A =$	$\sqrt{1-\sin^2 A}$	$\cos A$	$\dfrac{1}{\sqrt{1+\tan^2 A}}$	$\dfrac{\cot A}{\sqrt{1+\cot^2 A}}$	$\dfrac{1}{\sec A}$	$\dfrac{\sqrt{\operatorname{cosec}^2 A-1}}{\operatorname{cosec} A}$
$\tan A =$	$\dfrac{\sin A}{\sqrt{1-\sin^2 A}}$	$\dfrac{\sqrt{1-\cos^2 A}}{\cos A}$	$\tan A$	$\dfrac{1}{\cot A}$	$\sqrt{\sec^2 A-1}$	$\dfrac{1}{\sqrt{\operatorname{cosec}^2 A-1}}$
$\cot A =$	$\dfrac{\sqrt{1-\sin^2 A}}{\sin A}$	$\dfrac{\cos A}{\sqrt{1-\cos^2 A}}$	$\dfrac{1}{\tan A}$	$\cot A$	$\dfrac{1}{\sqrt{\sec^2 A-1}}$	$\sqrt{\operatorname{cosec}^2 A-1}$
$\sec A =$	$\dfrac{1}{\sqrt{1-\sin^2 A}}$	$\dfrac{1}{\cos A}$	$\sqrt{1+\tan^2 A}$	$\dfrac{\sqrt{1+\cot^2 A}}{\cot A}$	$\sec A$	$\dfrac{\operatorname{cosec} A}{\sqrt{\operatorname{cosec}^2 A-1}}$
$\operatorname{cosec} A =$	$\dfrac{1}{\sin A}$	$\dfrac{1}{\sqrt{1-\cos^2 A}}$	$\dfrac{\sqrt{1+\tan^2 A}}{\tan A}$	$\sqrt{1+\cot^2 A}$	$\dfrac{\sec A}{\sqrt{\sec^2 A-1}}$	cosec

$$\therefore\ \sin C+\sin A=\frac{3}{\sqrt{13}}+\frac{2}{\sqrt{13}}=\frac{5}{\sqrt{13}}$$

when $AB=2$, $BC=5-2=3$

$$\therefore\ \sin C=\frac{AB}{AC}=\frac{2}{\sqrt{13}},\ \sin A=\frac{BC}{AC}=\frac{3}{\sqrt{13}}$$

$$\therefore\ \sin C+\sin A=\frac{2}{\sqrt{13}}+\frac{3}{\sqrt{13}}=\frac{5}{\sqrt{13}}$$

Exercise 2

1. In a right-angled triangle ABC, hypotenuse $AB = 10$ cm, base $BC = 8$ cm and perpendicular $AC = 6$ cm. Find the values of sine and tangent of $\angle ABC$.

2. In a right-angled triangle, one of the acute angles is α. If $\tan\alpha=\frac{15}{8}$, find the values of $\sin\alpha$ and $\sec\alpha$.

3. Given that $\cot A=\frac{4}{7'5}$, find the values of $\cos A$ and $\operatorname{cosec} A$.

4. In an isosceles triangle ABC, $AB = AC = 10$ cm and the length of the bisector of the angle contained by two *equal* sides is 12 cm. Find the values of sine, cosine and tangent of any one of the base angles.

6. If $\sin\theta=\frac{3}{5}$, what is the value of $\frac{\operatorname{cosec}\theta}{1+\cot\theta}$?

7. If $\tan\theta=\frac{8}{15}$, show that $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}}=\frac{3}{5}$.

8. (*i*) If $\sec\theta+\tan\theta=2$, wht is the value of $\sec\theta-\tan\theta$?
   (*ii*) If $\operatorname{cosec}\theta-\cot\theta=\sqrt{2}-1$, what is the value of $\operatorname{cosec}\theta+\cot\theta$?
   (*iii*) If $\sin\theta+\cos\theta=1$, what is the value of $\sin\theta\cos\theta$?
   (*iv*) If $\tan\theta+\cot\theta=2$, what is the value of $\tan\theta-\cot\theta$?
   (*v*) If $\sin\theta-\cos\theta=\frac{7}{13}$, what is the value of $\sin\theta+\cos\theta$?
   (*vi*) If $\sin\theta\cos\theta=\frac{1}{2}$, what is the value of $\sin\theta-\cos\theta$?
   (*vii*) If $\sec\theta-\tan\theta=\frac{1}{\sqrt{3}}$, find the values of $\sec\theta$ and $\tan\theta$.

(*viii*) If $\operatorname{cosec}\theta + \cot\theta = \sqrt{3}$, find the values of $\operatorname{cosec}\theta$ and $\cot\theta$.

(*ix*) If $\frac{\sin\theta + \cos\theta}{\sin\theta - \cos\theta} = 7$, what is the value of $\tan\theta$?

(*x*) If $\frac{\operatorname{cosec}\theta + \sin\theta}{\operatorname{cosec}\theta - \sin\theta} = \frac{5}{3}$. what is the value of $\sin\theta$?

(*xi*) If $\sec\theta + \cos\theta = \frac{5}{2}$, what is the value of $\sec\theta - \cos\theta$?

(*xii*) What is the value of $\sin^2\theta + \cos^2\theta$? Using that value, find the value of $\tan\theta$ from the relation $5\sin^2\theta + 4\cos^2\theta = \frac{9}{2}$.

(*xiii*) If $\tan^2\theta + \cot^2\theta = \frac{10}{3}$, find the values of $\tan\theta + \cot\theta$ and $\tan\theta - \cot\theta$. Hence find the value of $\tan\theta$.

(*xiv*) If $\sec^2\theta + \tan^2\theta = \frac{13}{12}$, what is the value of $\sec^4\theta - \tan^4\theta$?

(*xv*) $\triangle PQR$ is right-angled at Q. $PR = \sqrt{5}$ and $PQ - RQ = 1$. Find the value of $\cos P - \cos R$.

(*xvi*) $\triangle XYZ$ is right-angled at Y. $XY = 2\sqrt{6}$ and $XZ - YZ = 2$. Find the value of $\sec X - \tan X$.

9. If $\sin\alpha = \frac{5}{13}$, show that $\tan\alpha + \sec\alpha = 1{\cdot}5$.

10. If $\tan\alpha = \frac{n}{m}$, find the values of $\sin A$ and $\sec A$.

11. If $\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$, show that $x\sin\theta = y\cos\theta$.

12. If $\sin\alpha = \frac{a^2-b^2}{a^2+b^2}$, show that $\cot\alpha = \frac{2ab}{a^2-b^2}$.

(In this exercise, each of the angles involved in given trigonometrical ratios and the trigometrical ratios to be found out in acute.)

Values of trigonometrical ratios of different angles:

In the earlier part of this chapter, you have seen that the values of trigonometrical ratios with respect to equal acute angles of two or more similar right-angled triangles are the same, whatever be the lengths of their sides.

$\triangle ABC$ and $\triangle DEF$ are two similar right-angled triangles in which $\angle ABC = \angle DEF = \theta$.

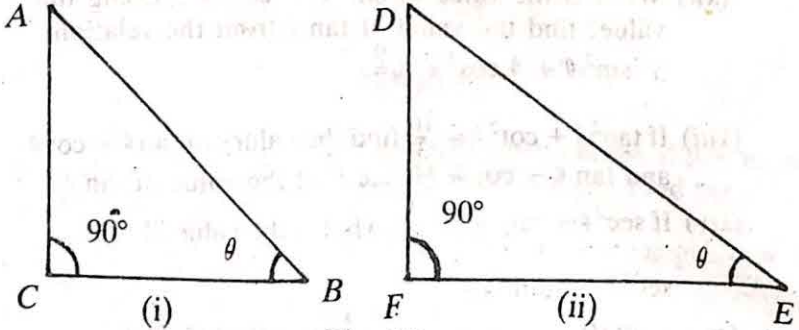


Fig. 22

Figures show that their sides are not equal. But you know $\frac{AC}{AB}=\frac{DF}{DE}$, that is, $\sin \angle ABC = \sin \angle DEF = \sin\theta$ and $\frac{AC}{BC} = \frac{DF}{EF}$, that is, $\tan \angle ABC = \tan \angle DEF = \tan\theta$.

Similarly, other trigonometrical ratios in reference to these two triangles will have the same value.

Hence it is clear that for an acute angle of any magnitude, we can find its trigonometrical ratios by drawing a right-angled triangle with that angle as one of its angles. For example,

(i) An angle of 15° is drawn at the end point *B* of a line segment *BC* of length 5 cm. Now a perpendicular *CA* is

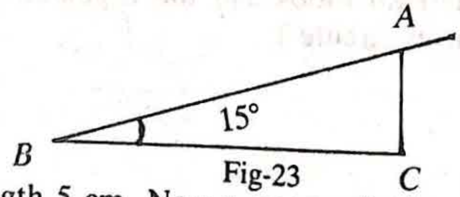


Fig-23

drawn on BC at the other end point C to meet the other arm of the angle at A. Thus we have a right-angled triangle ABC with $\angle ABC = 15^\circ$ and base $BC = 5$ cm. On measurement with a scale we will get $AB = 5{\cdot}2$ cm. (approx.) and $AC = 1{\cdot}3$ cm. (approx.).

Then we will have

$$\sin \angle ABC = \sin 15^\circ = \frac{AC}{AB} = \frac{1{\cdot}3 \text{ cm}}{5{\cdot}2 \text{ cm}} \text{(approx.)} = 0{\cdot}25 \text{ (approx.)}$$

$$\cos \angle ABC = \cos 15^\circ = \frac{BC}{AB} = \frac{5 \text{ cm}}{5{\cdot}2 \text{ cm}} \text{(approx.)} = 0{\cdot}96 \text{ (approx.)}$$

$$\tan \angle ABC = \tan 15^\circ = \frac{AC}{BC} = \frac{1{\cdot}3 \text{ cm}}{5 \text{ cm}} \text{(approx.)} = 0{\cdot}26 \text{ (approx.)}$$

Similarly, cosec 15°, sec 15°, cot 15° can be found out. Find yourself.

(ii) An angle of 20° is drawn at the end point E of the line segment EF of length 6 cm. At its other end point F, a perpendicular FD is drawn on it to meet the other arm of the angle at D. A right-angled triangle DEF is thus formed with $\angle DEF = 20^\circ$ and base $EF = 6$ cm.; now on measurement with a scale we will get $DE = 6{\cdot}4$ cm. (approx.) and $DF = 2{\cdot}2$ cm. (approx).

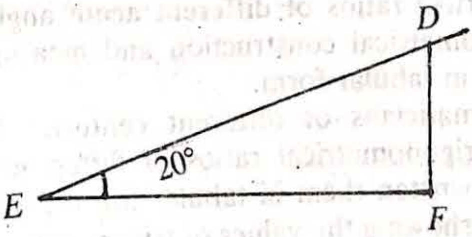


Fig. 24

Then we will have

$$\sin \angle DEF = \sin 20^\circ = \frac{DF}{DE} = \frac{2{\cdot}2 \text{ cm}}{6{\cdot}4 \text{ cm}} \text{(approx.)} = 0{\cdot}34 \text{ (approx.)}$$

$$\cos \angle DEF = \cos 20° = \frac{EF}{DE} = \frac{6 \text{ cm}}{6{\cdot}4 \text{ cm}} \text{(approx.)} = 0{\cdot}94 \text{ (approx.)}$$

$$\tan \angle DEF = \tan 20° = \frac{DF}{EF} = \frac{2{\cdot}2 \text{ cm}}{6 \text{ cm}} \text{(approx.)} = 0{\cdot}37 \text{ (approx.)}$$

Similarly, we can find the values of cosec 20°, sec 20° and cot 20°. You try it at home.

So you see, we can find trigonometrical ratios of any acute angle by geometrical construction and measurement. Now if we write those values in the tabular form as shown below, then, as is found necessary, we can have those values from the table to solve a problem.

Measure of angle	Trigonometrical ratios					
	sin	cos	tan	cosec	sec	cot
15°	0·25 approx.	0·96 approx.	0·26 approx.	4·0 approx.	1·04 approx.	3·85 approx.
20°	0·34 approx.	0·94 approx.	0·37 approx.	2·94 approx.	1·06 approx.	2·70 approx.

In this way you should try to find the values of trigonometrical ratios of different acute angles with the help of geometrical construction and measurement and write them in tabular form.

Mathematicians of different centuries have found values of trigonometrical ratios of different angles and have often written them in tabular form. At present, we have tables showing the values of trigonometrical ratios of all angles from 0° to 90°. You will have to use those tables when you will solve harder problems in higher classes.

But there are some angles of such measures between 0° and 90°, the values of trigonometrical ratios of which we can determine by geometrical reasoning. We need not take the trouble of geometrical construction and measurement thereof. Those angles are 0°, 30°, 45°, 60° and 90°. They are known as standard angles.

The methods of finding the values of trigonometrical ratios of these standard angles are given below. The trigonometrical problems that you will face at the present stage will be confined within these standard angles only. So it is essential for you to know the values of trigonometrical ratios corresponding to these standard angles. You will also have to commit them to memory through repeated practice, so that you can use them correctly and quickly if a situation arises.

Trigonometrical ratios of standard angles :

It has been said earlier that 0°, 30°, 45°, 60° and 90° are standard angles, as we can find the values of their trigonometrical ratios with simple geometrical reasoning. The methods are given below:

Trigonometrical ratios of 45° :

In the right-angled triangle ABC, $\angle ACB = 45°$; we are to find the values of trigonometrical ratios of this angle.

$\because \angle ACB = 45°$,

$\therefore \angle CAB = 45°$,

$\therefore CB = AB$.

Let $CB = AB = a$

$\therefore CA^2 = CB^2 + AB^2$

$= a^2 + a^2 = 2a^2$

$\therefore CA = \sqrt{2a^2} = a\sqrt{2}$.

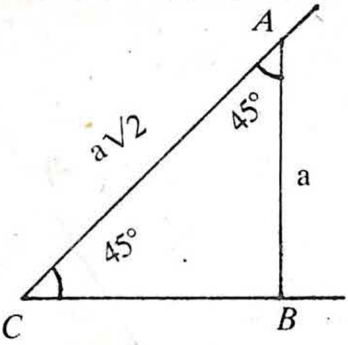


Fig. 25

So here, hypotenuse $CA = a\sqrt{2}$, perpendicular $AB = a$ and base $BC = a$.

$$\text{So, } \sin 45° = \frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\operatorname{cosec} 45° = \frac{\text{hypotenuse}}{\text{perpendicular}} = \frac{a\sqrt{2}}{a} = \sqrt{2}$$

$$\cos 45^\circ = \frac{\text{base}}{\text{hypotenuse}} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\sec 45^\circ = \frac{\text{hypotenuse}}{\text{base}} = \frac{a\sqrt{2}}{a} = \sqrt{2}$$

$$\tan 45^\circ = \frac{\text{perpendicular}}{\text{base}} = \frac{a}{a} = 1$$

$$\cot 45^\circ = \frac{\text{base}}{\text{perpendicular}} = \frac{a}{a} = 1.$$

Values of trigonometrical ratios of 60° :

In the right-angled triangle AOB, $\angle AOB = 60^\circ$. We are to find the values of the trigonometrical ratios of this angle.

Construction: Let OB be produced to the point C, such that $OB = BC$. A, C are joined.

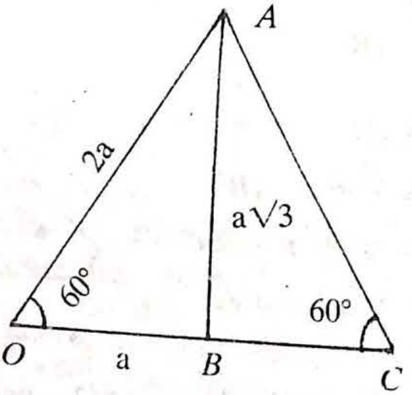


Fig-26

Now in $\triangle AOB$ and ABC,

$OB = BC$

AB is common

and included $\angle ABO$ = included $\angle ABC$ (both being right angles).

$\therefore$ Two triangles are congruent.

$\therefore \angle AOB = 60^\circ = \angle ACB$

$\therefore \triangle AOC$ is an equilateral triangle.

Let $OA = 2a$, $\therefore OC = 2a$.

But $OB = \frac{1}{2}\ OC = \frac{1}{2} \times 2a = a.$

$\therefore$ From $\triangle AOB$ we get $AB^2 = OA^2 - OB^2$

$= 4a^2 - a^2 = 3a^2$

$\therefore\ AB = \sqrt{3a^2} = a\sqrt{3}.$

Therefore, in $\triangle AOB$, with respect to $\angle AOB$ hypotenuse $OA = 2a$, perpendicular $AB = a\sqrt{3}$, base $OB = a$

$$\text{So, } \sin 60° = \frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$$

$$\operatorname{cosec} 60° = \frac{\text{hypotenuse}}{\text{perpendicular}} = \frac{2a}{a\sqrt{3}} = \frac{2}{\sqrt{3}}$$

$$\cos 60° = \frac{\text{base}}{\text{hypotenuse}} = \frac{a}{2a} = \frac{1}{2}$$

$$\sec 60° = \frac{\text{hypotenuse}}{\text{base}} = \frac{2a}{a} = 2$$

$$\tan 60° = \frac{\text{perpendicular}}{\text{base}} = \frac{a\sqrt{3}}{a} = \sqrt{3}$$

$$\cot 60° = \frac{\text{base}}{\text{perpendicular}} = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Values of trigonometrical ratios of 30° :

In the right-angled triangle AOB, $\angle BOA = 30°$.

We are to find the values of the trigonometrical ratios of this angle.

Construction: Let AB be produced to the point C, such that $AB = BC$. O, C are joined.

Now, in $\triangle^s$ AOB and BOC,

$AB = BC$,

OB is common

and included $\angle ABO$ = included $\angle CBO$ (both being right angles).

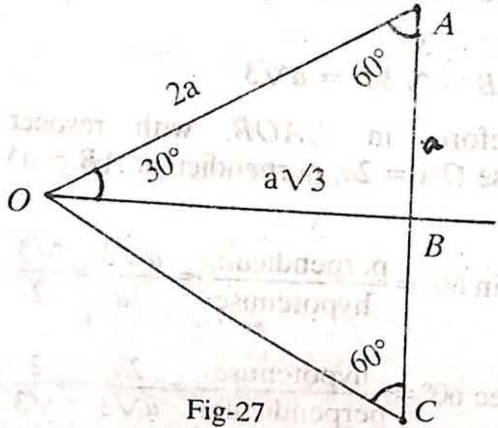


Fig-27

$\therefore$ The two triangles are congruent.
$\therefore \angle OAB = 60° = \angle OCB$
$\therefore \triangle AOC$ is an equilateral triangle.
Let $OA = 2a$, $\therefore AC = 2a$
$\therefore AB = \frac{1}{2} AC = \frac{1}{2} \times 2a = a.$
In the right-angled triangle AOB,
$OB^2 = OA^2 - AB^2 = 4a^2 - a^2 = 3a^2$
$\therefore OB = \sqrt{3a^2} = a\sqrt{3}.$

Then in $\triangle AOB$, with respect to $\angle AOB$ hypotenuse $OA=2a$, perpendicular $AB=a$, and base $OB=a\sqrt{3}$.

Therefore, $\sin 30° = \frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{a}{2a} = \frac{1}{2}$

$$\operatorname{cosec} 30° = \frac{\text{hypotenuse}}{\text{perpendicular}} = \frac{2a}{a} = 2$$

$$\cos 30° = \frac{\text{base}}{\text{hypotenuse}} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$$

$$\sec 30° = \frac{\text{hypotenuse}}{\text{base}} = \frac{2a}{a\sqrt{3}} = \frac{2}{\sqrt{3}}$$

$$\tan 30^\circ = \frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\cot 30^\circ = \frac{\text{hypotenuse}}{\text{perpendicular}} = \frac{a\sqrt{3}}{a} = \sqrt{3}$$

Remark: You can also find the values of trigonometrical ratios of 60° and 30° by *(i)* drawing an angle $\angle ABD = 30^\circ$ to meet the hypotenuse at the point D or *(ii)* joining D, the mid-point of the hypotenuse OA with B and using the property that OA is the diameter of the circumcircle of $\triangle AOB$. Try at home.

Values of trigonometrical ratios of 90° :

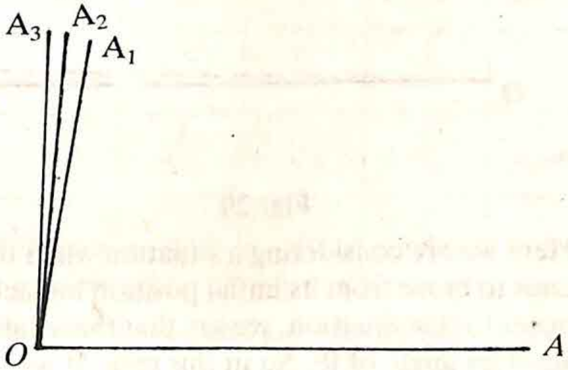


Fig. 28

Here we are considering a situation when the rotating ray tends to form an angle of 90° and from any point on the ray a perpendicular is drawn on the initial position of ray to get a right-angled triangle. Figure shows that when the ray tends to produce an angle of 90°, the hypotenuse approaches perpendicular and base tends to zero. Therefore, in this case, we may take perpendicular = hypotenuse and base = 0.

$$\text{Therefore, } \sin 90^\circ = \frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{\text{perpendicular}}{\text{perpendicular}} = 1$$

$$\operatorname{cosec} 90^\circ = \frac{\text{hypotenuse}}{\text{perpendicular}} = \frac{\text{perpendicular}}{\text{perpendicular}} = 1$$

$$\cos 90^\circ = \frac{\text{base}}{\text{hypotenuse}} = \frac{0}{\text{hypotenuse}} = 0$$

$$\cot 90^\circ = \frac{\text{base}}{\text{perpendicular}} = \frac{0}{\text{perpendicular}} = 0$$

N.B.: sec 90° and tan 90° are undefined.

Values of trigonometrical ratios of 0° :

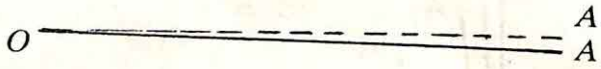


Fig. 29

Here we are considering a situation when the rotating ray tends to move from its initial position but actually does not more. In this situation, we say that the rotating ray has produced an angle of 0°. So in this case, if we try to get a right-angled triangle by drawing a perpendicular we may assume that, perpendicular=0 and base= hypotenuse.

$$\text{So, } \sin 0^\circ = \frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{0}{\text{hypotenuse}} = 0$$

$$\cos 0^\circ = \frac{\text{base}}{\text{hypotenuse}} = \frac{\text{base}}{\text{base}} = 1$$

$$\tan 0^\circ = \frac{\text{perpendicular}}{\text{base}} = \frac{0}{\text{base}} = 0.$$

N.B.: cosec 0° and cot 0° are undefined.

Table showing the values of trigonometrical ratios of standard angles:

Angles→ Ratios ↓	$0°$ or 0^c	$30°$ or $\frac{\pi}{6}$	$45°$ or $\frac{\pi}{4}$	$60°$ or $\frac{\pi}{3}$	$90°$ or $\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Undefined
cosec	Undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Undefined
cot	Undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

N.B.: Values of $\sin\theta$ and $\cos\theta$ lie between 0 and 1.

Example 1: At one end point A of the line segment AB, $\angle BAX=22{\cdot}5°$ is drawn and at the other end point B a perpendicular is drawn on it to meet AX at C. Now if $AB=7$ cm. and $BC=2{\cdot}9$ cm, find sine, cosine and tangent of $22{\cdot}5°$ and $67{\cdot}5°$ (correct upto two decimal places).

In the right-angled triangle ABC, $\angle BAC=22{\cdot}5°$

$\therefore\ \angle ACB=90°-22{\cdot}5°=67{\cdot}5°$

$AC=\sqrt{AB^2+BC^2}$

$=\sqrt{7^2+2{\cdot}9^2}$

$=7{\cdot}6$

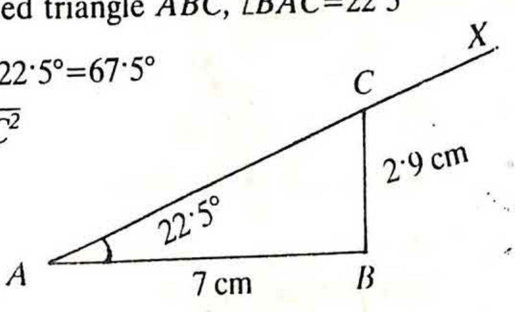


Fig. 30

$\therefore\ AB=7$ cm., $AC=7.6$ cm. and $BC=2{\cdot}9$ cm.

Now, with respect to $\angle BAC=22{\cdot}5^\circ$, AB=base, BC=perpendicular and AC=hypotenuse, and with respect to $\angle ACB=67{\cdot}5^\circ$

BC=base, AB=perpendicular and AC=hypotenuse

$$\therefore \sin 22{\cdot}5^\circ = \frac{2{\cdot}9 \text{ cm}}{7{\cdot}6 \text{ cm}} \text{ (approx)} = 0{\cdot}38 \text{ (approx)}$$

$$\sin 67{\cdot}5^\circ = \frac{7 \text{ cm}}{7{\cdot}6 \text{ cm}} \text{ (approx)} = 0{\cdot}92 \text{ (approx)}$$

$$\cos 22{\cdot}5^\circ = \frac{7 \text{ cm}}{7{\cdot}6 \text{ cm}} \text{ (approx)} = 0{\cdot}92 \text{ (approx)}$$

$$\cos 67{\cdot}5^\circ = \frac{2{\cdot}9 \text{ cm}}{7{\cdot}6 \text{ cm}} \text{ (approx)} = 0{\cdot}38 \text{ (approx)}$$

$$\tan 22{\cdot}5^\circ = \frac{2{\cdot}9 \text{ cm}}{7 \text{ cm}} \text{ (approx)} = 0{\cdot}41 \text{ (approx)}$$

$$\tan 67{\cdot}5^\circ = \frac{7 \text{ cm}}{2{\cdot}9 \text{ cm}} \text{ (approx)} = 2{\cdot}41 \text{ (approx)}$$

Example 2: A boat crosses a stream 188 metre wide, in 4 minutes. If the speed of the boat is 3 km/hr. and being drifted by the current, the boat moves in a direction making 70° with this bank of the stream, prove that, $\cos 20^\circ=0{\cdot}94$.

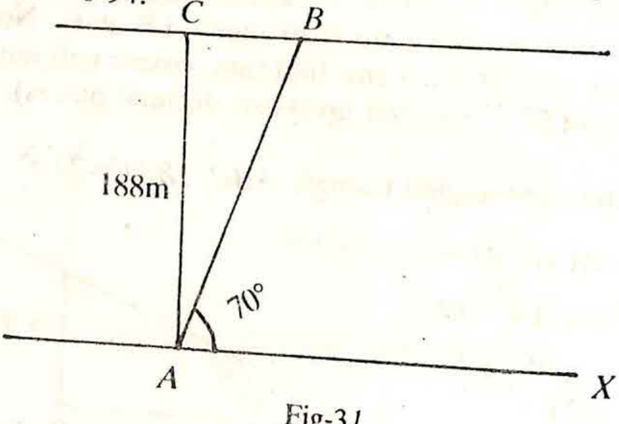


Fig-31

Let us suppose that the boat starts from A and reaches B in 4 minutes.

$\angle XAB=70^\circ$ and AC=width of the stream=188 m.

$\because \angle XAC=90$
.007 and $\angle XAB=70^\circ$,

$\therefore \angle BAC=90^\circ-70^\circ=20^\circ$

$\because$ the boat covers distance AB in 4 minutes moving with a speed of 3 km/hr,

$$AB=3 \text{ km} \times \frac{4}{60}= 200 \text{ m.}$$

$$\therefore \cos \angle CAB=\cos 20^\circ = \frac{AC}{AB}=\frac{188}{200}= 0{\cdot}94 \text{ (proved).}$$

Example 3: Show that $\sin^2 30^\circ+\cos^2 30^\circ=1$.

$$\sin^2 30^\circ+\cos^2 30^\circ= \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4}+\frac{3}{4}= 1$$

Example 4: Find the value of $\sin^2 30^\circ+4 \cot^2 45^\circ-\sec^2 60^\circ$

$\sin^2 30^\circ+4 \cot^2 45^\circ-\sec^2 60^\circ$

$= \left(\frac{1}{2}\right)^2 + 4.(1)^2-(2)^2=\frac{1}{4}+4-4 = \frac{1}{4}$

Example 5: AD is a median of an equilateral triangle ABC. Prove that, $\sin \angle BAD=\cos \angle ABD$.

Since $\triangle ABC$ is equilateral and AD is a median,

$\therefore AD$ is perpendicular on BC and it bisects $\angle BAC$.

$\therefore \angle BAD=30^\circ$ sin and $\angle ABD=60^\circ$

$\therefore \sin \angle BAD = 30^\circ$

$= \frac{1}{2} =\cos 60^\circ = \cos \angle ABD$.

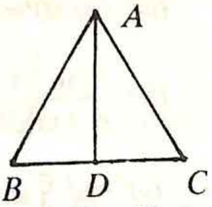


Fig. 30 *A*

Example 6: Prove that $\cos 60^\circ =\dfrac{1-\tan^2 30^\circ}{1+\tan^2 30^\circ}$ $\text{Cos } 60^\circ=\frac{1}{2}$

$$\frac{1-\tan^2 30°}{1+\tan^2 30°}=\frac{1-\frac{1}{(\sqrt{3})}^{2}}{1+\frac{1}{(\sqrt{3})}^{2}}=\frac{1-\frac{1}{3}}{1+\frac{1}{3}}=\frac{2}{3}\times\frac{3}{4}=\frac{1}{2}=\cos 60°$$

$$\therefore\ \cos 60°=\frac{1-\tan^2 30°}{1+\tan^2 30°}$$

Exercise 3

1. In an isosceles triangle ABC, $AB=AC=10$ cm and $BC=16$ cm. If $\angle BAC=106°$, find the values of the following trigonometrical ratios (correct up to two decimal places).

Measure of angle	Trigonometrical ratios					
	sin	cos	tan	cosec	sec	tan
37°						
53°						

2. Show that :

(i) $\frac{\sin 60°}{\cos 60°}=\tan 60°$ (ii) $\sin^2 45°+\cos^2 45°=1$

(iii) $\cos 60°=\cos^2 30°-\sin^2 30°$

(iv) $\frac{2\tan 30°}{1-\tan^2 30°}=\sqrt{3}$

(v) $\tan^2\frac{\pi}{4}\sin\frac{\pi}{3}\tan\frac{\pi}{3}\tan^2\frac{\pi}{3}=1\frac{1}{2}$

(vi) $\sin\frac{\pi}{3}\tan\frac{\pi}{6}+\sin\frac{\pi}{2}\cos\frac{\pi}{3}=2\sin^2\frac{\pi}{4}$

3. Find the values of:

(i) $\sin^2 45° - \operatorname{cosec}^2 60° + \sec^2 30°$

(ii) $\sec^2 45° - \cot^2 45° - \sin^2 30° - \sin^2 60°$

(iii) $3 \tan^2 45° - \sin^2 60° - \frac{1}{3} \cot^2 30° - \frac{1}{8} \sec^2 45°$

(iv) $\frac{4}{3} \cot^2 30° + 3 \sin^2 60° - 2 \operatorname{cosec}^2 60° - \frac{3}{4} \tan^2 30°$

(v) $\dfrac{\frac{1}{3} \cos 30°}{\frac{1}{2} \sin 45°} + \dfrac{\tan 60°}{\cos 30°}$

(vi) $\sec^2 60° - \cot^2 30° - \dfrac{2 \tan 30° \operatorname{cosec} 60°}{1 + \tan^2 30°}$

(vii) $4 \sin^2 45° + \tan^2 60° + \operatorname{cosec}^2 30°$

(viii) $\dfrac{\tan 60° - \tan 30°}{1 + \tan 60° \tan 30°} + \cos 60° \cos 30° + \sin 60° \sin 30°$

(ix) $\cot^2 30° - 2\cos^2 60° - \frac{3}{4} \sec^2 45° - 4 \sin^2 30°$

(x) $\dfrac{\tan^2 60° + \tan 30°}{1 + \tan 60° \tan 30°} + \cos 60° \cos 30° - \sin 60° \sin 30°$

(xi) $\dfrac{1 + \tan^2 30°}{1 - \tan^2 45°} + \operatorname{cosec}^2 60° - \cos^2 45° + \sin^2 45° + \dfrac{1 + \cot^2 60°}{1 - \cot^2 60°}$

(xii) $\dfrac{1 - \sin^2 30°}{1 + \sin^2 45°} \times \dfrac{\cos^2 60° + \cos^2 30°}{\operatorname{cosec}^2 90° - \cot^2 90°} \div (\sin 60° \tan 30°)$

4. Prove that—

(i) $\tan^2 60° - 2 \sin 60° = 3 - \cot 30°$

(ii) $\sqrt{\dfrac{1 + \cos 30°}{1 - \cos 30°}} = \sec^2 45° - \cot^2 45° = \sec 60°$

(iii) $\dfrac{2 \tan^2 30°}{1 - \tan^2 30°} + \sec^2 45° - \cot^2 45° = \sec 60°$

5. (i) BD is a median of an equilateral triangle ABC. Prove that $\tan \angle ABD = \cot \angle BAD$.

(ii) In an isosceles triangle ABC, $AB = AC$ and $\angle BAC = 90°$. If the bisector of BAC meets BC at the point D, prove that

$$\frac{\sec \angle ACD}{\sin \angle CAD} = \operatorname{cosec}^2 \angle CAD$$

6. If $x \sin 45° \cos 45° \tan 60° = \tan^2 45° - \cos 60°$, find the value of x.

7. If $x \sin 60° \cos^2 30° = \frac{\tan^2 45° \sec 60°}{\operatorname{cosec} 60°}$, find the value of x.

8. If $x^2 = \sin^2 30° + 4 \cot^2 45° - \sec^2 60°$, find the value of x.

9. To locate Habibpur High School one has to walk $530\sqrt{2}$ metres in the north-west direction from a point on No. 3 National Highway, which runs north-south. What is the distance of the school from the National Highway?

CHAPTER III

TRIGONOMETRICAL RATIOS OF COMPLEMENTARY ANGLES

You know from geometry that one angle is said to be complement of another, when the sum of the measures of two angles is equal to 90°. For example, as $30° + 60° = 90°$, 60° is complement of 30° and 30° is complement of 60°. Thus 53° is complement of 37°, 67·5° is complement of 22·5° etc. In general, if $\theta°$ be an angle then its complementary angle will be $90°-\theta°$.

Let us now see how we can find the trigonometrical ratios of $90°-\theta°$, if we know those of $\theta°$.

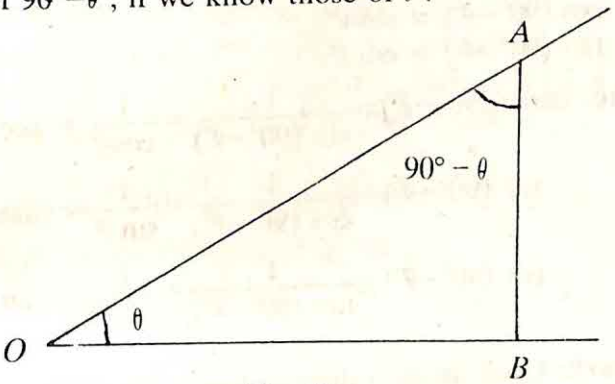


Fig. 31

In the adjoining figure, in the right-angled triangle, AOB, $\angle AOB = \theta$; then its complementary angle $\angle OAB$ is $90°-\theta$.

Can you name the hypotenuse, perpendicular and base with respect to each of these two acute angles?

With respect to $\theta°$	With respect to $(90°-\theta°)$
Hypotenuse $= OA$	Hypotenuse $= OA$
Perpendicular $= AB$	Perpendicular $= OB$
Base $= OB$	Base $= AB$

Now, in case of $(90°-\theta°)$,

$$\sin(90°-\theta°) = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{OB}{OA}$$

But in case of $\theta°$, $\dfrac{OB}{OA}=\dfrac{\text{Base}}{\text{Hypotenuse}}=\cos\theta°$;

In case of $(90°-\theta°)$, $\cos(90°-\theta°)=\dfrac{\text{Base}}{\text{Hypotenuse}}=\dfrac{AB}{OA}$

But in case of $\theta°$, $\dfrac{AB}{OA}=\dfrac{\text{Perpendicular}}{\text{Hypotenuse}}=\sin\theta°$;

In case of $(90°-\theta°)$, $\tan(90°-\theta°)=\dfrac{\text{Perpendicular}}{\text{Base}}=\dfrac{OB}{AB}$

But in case of $\theta°$, $\dfrac{OB}{AB}=\dfrac{\text{Base}}{\text{Perpendicular}}=\cot\theta°$

From above discussion, we get

$$\sin(90°-\theta°)=\cos\theta°$$
$$\cos(90°-\theta°)=\sin\theta°$$
$$\tan(90°-\theta°)=\cot\theta°.$$

Therefore, $\operatorname{cosec}(90°-\theta°)=\dfrac{1}{\sin(90°-\theta°)}=\dfrac{1}{\cos\theta°}=\sec\theta°$

$$\sec(90°-\theta°)=\frac{1}{\cos(90°-\theta°)}=\frac{1}{\sin\theta°}=\operatorname{cosec}\theta°$$

$$\cot(90°-\theta°)=\frac{1}{\tan(90°-\theta°)}=\frac{1}{\cot\theta°}=\tan\theta°.$$

Remark: Look at the values of trigonometrical ratios of 22·5° and 67·5°, which you have found in Example 1. sin 22·5° = 0·38 = cos 67·5°; because 67·5° is complement to 22·5°. From the results obtained in that example find the values of cosecant, secant and cotangent of 22·5° and 67·5° and compare.

The values of trigonometrical ratios of 30° and 60° (which are complementary angles) are compared below. This will help you to have a clear understanding of the relations shown above

$\sin 30° = \cos 60° = \frac{1}{2}$ $\qquad$ $\operatorname{cosec} 30° = \sec 60° = 2$

$\cos 30° = \sin 60° = \dfrac{\sqrt{3}}{2}$ $\qquad$ $\sec 30° = \operatorname{cosec} 60° = \dfrac{2}{\sqrt{3}}$

$$\tan 30^\circ = \cot 60^\circ = \frac{1}{\sqrt{3}} \qquad \cot 30^\circ = \tan 60^\circ = \sqrt{3}.$$

Again from these formulae we get

$$\sin 45^\circ = \cos 45^\circ = \frac{1}{\sqrt{2}} \qquad \sin 90^\circ = \cos 0^\circ = 1$$

$$\operatorname{cosec} 45^\circ = \sec 45^\circ = \sqrt{2} \qquad \cos 90^\circ = \sin 0^\circ = 0$$

$$\tan 45^\circ = \cot 45^\circ = 1.$$

Example 1: A and B are two complementary angles. Prove that,

$$(\sin A + \sin B)^2 = 1 + 2 \sin A \cos A.$$

$\because$ A and B are complementary angles.

$\therefore$ $\sin B = \sin (90^\circ - A) = \cos A$

$$\therefore (\sin A + \sin B)^2 = (\sin A + \cos A)^2$$
$$= \sin^2 A + \cos^2 A + 2 \sin A \cos A$$
$$= 1 + 2 \sin A \cos A.$$

Example 2: Show that, $\sin {}^2 21^\circ + \sin^2 69^\circ = 1$

$$\sin 79^\circ = \sin (90^\circ - 21^\circ) = \cos 21^\circ$$

$$\therefore \sin^2 21^\circ + \sin^2 69^\circ = \sin^2 21^\circ + \cos^2 21^\circ = 1.$$

Example 3: Show that, $\tan 15^\circ + \tan 75^\circ = \dfrac{\sec^2 15^\circ}{\sqrt{\sec^2 15^\circ - 1}}$

Proof: $\tan 15^\circ + \tan 75^\circ = \tan 15^\circ + \tan (90^\circ - 15^\circ)$

$$= \tan 15^\circ + \cot 15^\circ = \tan 15^\circ + \frac{1}{\tan 15^\circ}$$

$$= \frac{\tan^2 15^\circ + 1}{\tan 15^\circ} \qquad = \frac{\sec^2 15^\circ}{\sqrt{\sec^2 15^\circ - 1}}$$

Exercise 4

1. If α and ß are two complementary angles, prove that—

(a) $\sin^2 \quad + \sin^2\beta = 1$ (b) $\cot ß + \cos ß = \frac{\cos ß}{\cos \alpha}(1+\sin ß)$

(c) $\frac{\sec \alpha}{\cos \alpha} - \cot^2 ß = 1$

2. If $\sin 17° = \frac{x}{y}$, show that

$$\sec 17° - \sin 73° = \frac{x^2}{y\sqrt{y^2 - x^2}}$$

3. Prove that, $\sec^2 12° - \frac{1}{\tan^2 78°} = 1$

4. $\angle A + \angle B = 90°$; show that, $1 + \frac{\tan A}{\tan B} = \sec^2 A$.

5. Prove that,

$\operatorname{cosec}^2 22° . \cot^2 68° = \sin^2 22° + \sin^2 68° + \cot^2 68°$

6. If $\angle P + \angle Q = 90°$, show that,

$$\cos P = \sqrt{\frac{\sin P}{\cos Q} - \sin P \cos Q}$$

7. AOB is a diameter of a circle with centre O. C is a point on the circle. Join $A, C; B, C; O, C$ and prove that,

(i) $\tan \angle ABC = \cot \angle ACO$

(ii) $\sin^2 \angle BCO + \sin^2 \angle ACO = 1$

(iii) $\operatorname{cosec}^2 \angle CAB - 1 = \tan^2 \angle ABC$

8. $ABCD$ is a rectangle. Join A, C and prove that,

(i) $\tan \angle ACD = \cot \angle ACB$

(ii) $\tan^2 \angle CAD + 1 = \frac{1}{\sin^2 \angle BAC}$

CHAPTER IV

APPLICATION OF TRIGONOMETRICAL RATIOS IN PRACTICAL PROBLEMS

In our introductory note 'Why Trigonometry' we said that we could find easily heights, lengths of some objects or distances between two objects with the help of trigonometry, which could not be done so easily by direct measurement. Let us suppose that there is a steep high hill somewhere. We will have to find out its height. It is difficult to climb the hill; it may even be dangerous. But without taking the trouble and risk of climbing the hill, we can find its height standing on the ground with the help of trigonometry. Similarly, from one bank of a canal we can find the height of a palm tree on the other bank or the distance of the palm tree from this bank without crossing the canal. We can also find the heights of trees or monuments by measuring the lengths of their shadows. Even more than that, we can say from the earth how long the diameters of the Sun and the Moon are, how far they are from the Earth etc. Some of these are illustrated in the following examples.

Angle of elevation and angle of depression :

Let AB be a pole with A as the top. BP is the horizontal line through the foot of the pole (see figure).

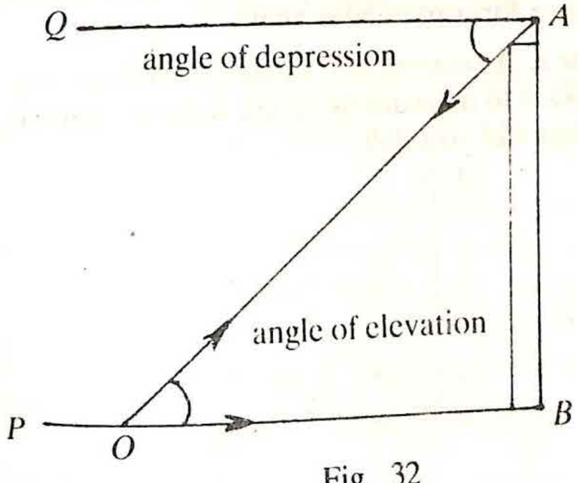


Fig. 32

Now let us suppose that O is a point on BP.

If an observer keeps his eyes at the point O and looks at the top of the pole he will be able to see the top of the pole at an angle BOA. This angle is called *angle of evlevation.* Therefore, the angle of elevation of the point A with respect to the point O is $\angle BOA$.

Again, it may happen that a man climbs up the pole, keeps his eyes at the point A and sees an object placed at the point O on the horizontal line. Then the angle at which the man will see the object at the point O is the *angle of depression* of the point O with respect to the point A. How can we get the angle of depression? We shall have to imagine a straight line AQ parallel to the straight line BP through A, that is, a horizontal line through the point A. The measure of the angle of depression will be $\angle QAO$

A point to be noted :

Here, $BO \parallel AQ$ and OA is the transversal. So $\angle AOB$, the angle of elevation $= \angle OAQ$, the angle of depression (being alternate angles). But even then they are to be indicated to solve problems.

Remark: It has already been mentioned that generally you may have to deal with standard angles. If a problem involves any other angle, the necessary trigonometrical ratios will be given. Of course, in higher classes you will have to use the tables of trigonomentrical ratios for finding values of those for a particular angle.

Example 1: A coconut tree stands on the bank of your pond. You want to measure its height without climbing up the tree. What will you do?

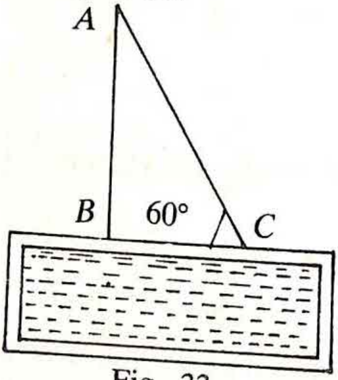


Fig. 33

Let AB represent the coconut tree. Its height is to be determined.

You proceed along the bank of the pond to a point which is at some distance from the foot of the tree. Then you find the angle of elevation of the top of the tree. You can now find the height of the tree using trigonometrical ratios.

Here, you suppose that you have moved along the bank of the pond to a point C, 12 metres apart from the point B, the foot of the coconut tree. The angle of elevation of the point A with respect to the point C is supposed to be 60°.

Now, if A, C are joined the right-angled triangle ABC will be obtained of which $\angle ACB = 60°$. With respect to this angle the base BC = 12 metres and the perpendicular AB = the height of the coconut tree = x metres (say)

$$\therefore \tan \angle ACB = \tan 60° = \frac{\text{perpendicular}}{\text{base}} = \frac{x}{12}$$

Also you get, from the table of standard angles,
$\tan 60° = \sqrt{3}$

$$\therefore \frac{x}{12} = \sqrt{3}$$

$$\begin{aligned} or,\ x &= 12\sqrt{3} = 12 \times 1{\cdot}732 \text{ (approx.)} \\ &= 20{\cdot}784 \text{ (approx.)} \end{aligned}$$

∴ the height of the coconut tree is 20·784 metres (approx.) or, 20 metres and 78.4 centimetres (approx.).

Example 2: A light post is bent at a point due to storm and the top of the light post meets the ground at a point 4 metres apart from the foot of the light post making an angle of the light post making an angle of 45° with the horizontal line through these two points. How long was the light post?

Let AB be the light post. It is bent at the point O and the top of the light post meets the ground at the point C.

$\therefore AO = OC$ and $AB = AO + OB$. Let $AO = OC =$ x metres.

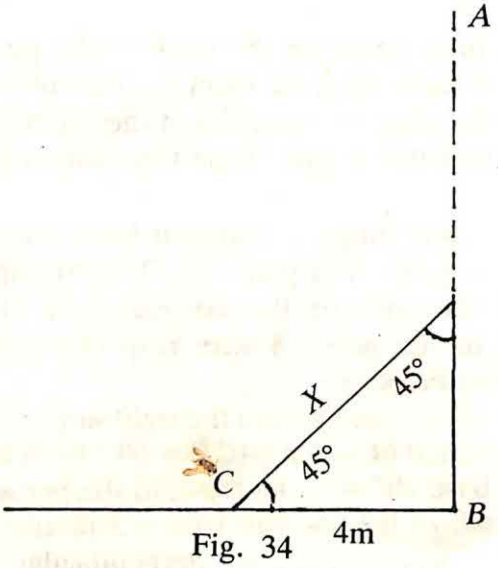


Fig. 34

From the figure, you see that in the right-angled $\triangle BCO$, $\angle BCO = 45°$, the base $BC = 4$ metres and the hypotenuse $OC = x$ metres.

Also see, since $\angle BCO = 45°$, therefore, the complementary angle $\angle BOC = 45°$

$\therefore BC = BO = 4$ metres.

Now, $\cos \angle BCO = \cos 45° = \dfrac{\text{base}}{\text{hypotenuse}} = \dfrac{4}{x}$.

Again we get from the table, $\cos 45° = \dfrac{1}{\sqrt{2}}$

$$\therefore \frac{4}{x} = \frac{1}{\sqrt{2}}$$

$$\therefore x = 4\sqrt{2} = 4 \times 1{\cdot}414 \text{ (approx.)}$$
$$= 5{\cdot}656 \text{ (approx.)}$$

$\therefore$ the height of the light post $AB = AO + OB =$ $=5{\cdot}656 + 4)$ metres (approx.) $= 9{\cdot}66$ metres (approx.). that is, 9 m 66 cm (approx.).

Example 3: A three-storied building stands on the bank of a canal. You observe from a corner of the roof of the building, the foot of a kilometre post just on the opposite bank. If the angle of depression of the foot of the post at your eye is 30° and the height of the building is 10 metres, what is the width of the canal?

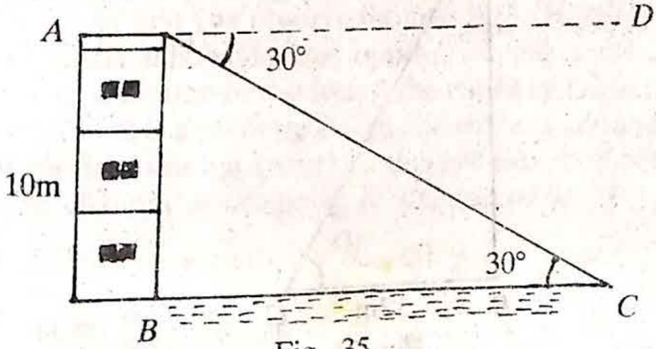


Fig. 35

Let A, B, C denote the corner of the roof of the building, the foot of the building vertically below the corner point and the foot of the kilometre post just on the opposite bank respectively. A right-angled triangle ABC is formed by joining these points.

Let AD be the horizontal line through A.

$\angle DAC$, the angle of depression $= \angle ACB = 30°$, and with respect to this angle perpendicular $AB = 10$ metres and base $BC =$ width of the canal $= x$ metres.

$$\therefore \tan 30° = \frac{AB}{BC} = \frac{10}{x} = \frac{1}{\sqrt{3}}$$

$\therefore x = 10\sqrt{3} = 10 \times 1{\cdot}732$ (approx.) $= 17{\cdot}32$ (approx.)

∴ The width of the canal is 17.32 m (approx.) or 17 m 32 cm (approx).

Example 4: When altitude of the Sun is 60°, the shadow of a palm tree is 12 m in length. What is the height of the palm tree?

Let AB denote the height of the palm tree and BC denote the length of the shadow and then

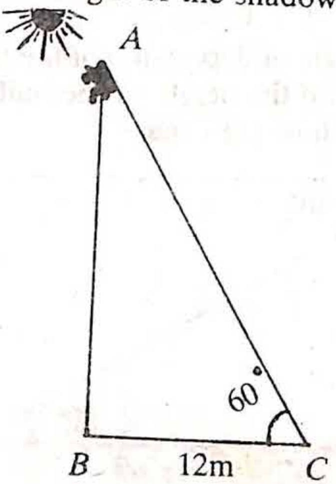


Fig 36

$\therefore \angle ACB = 60°$, $BC = 12$ m. Let, $AB = x$ metres.

Now, $\tan 60° = \frac{AB}{BC} = \frac{x}{12} = \sqrt{3}$

$$\therefore x = 12\sqrt{3} = 12 \times 1{\cdot}732 \text{ (approx.)}$$
$$= 20{\cdot}784 \text{ (approx.)}$$

$\therefore$ the height of the palm tree is 20·784 metres (approx).

Example 5: You are standing on a bank of a river. A factory is situated at the other bank. You are observing the

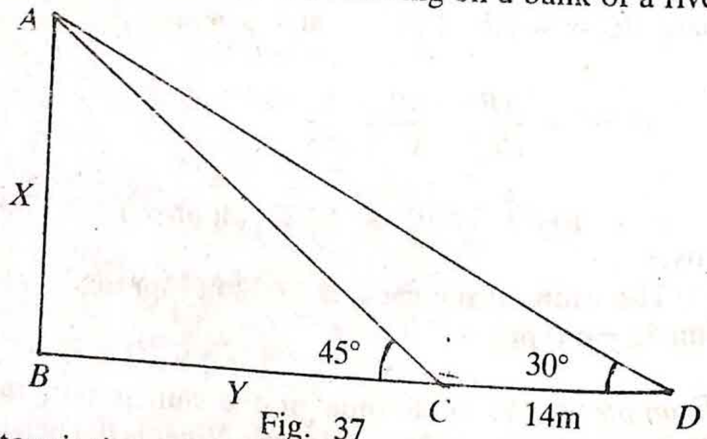


Fig. 37

upper part of the chimney from a point.

You see how the height of the chimney can be determined from this bank without crossing the river.

Let AB denote the height of the chimney. Let $AB = x$ metres. At first you observe the top of the chimney from a point C on this bank, just opposite to the point B. The angle of elevation of the top of the chimney at the point C is 45° and D is a point on BC produced, at a distance of 14 metres from this bank and the angle of elevation of the top of the chimney at the point D is found to be 30°.

Let $BC = y$ metres = breadth of the river.

$$\text{Then, } \frac{AB}{BC} = \frac{x}{y} = \tan 45^\circ = 1 \qquad ..(i)$$

$$\frac{AB}{BD} = \frac{x}{y+14} = \tan 30^\circ = \frac{1}{\sqrt{3}} \qquad ..(ii)$$

Now from equation no. *(i)* we get, $x=y$ and substituting x for y and in equation no. *(ii)* we get,

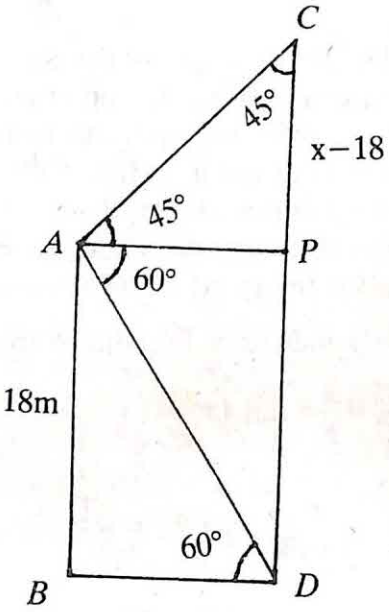
$$\frac{x}{x+14} = \frac{1}{\sqrt{3}}$$

$$\text{or, } x\sqrt{3} = x+14$$
$$\text{or, } x\sqrt{3} - x = 14$$
$$\text{or, } x(\sqrt{3} - 1) = 14$$

$$\text{or, } x = \frac{14}{\sqrt{3} - 1} = \frac{14(\sqrt{3}+1)}{2}$$
$$= 7\,(1{\cdot}732+1) \text{ (approx.)}$$
$$= 19{\cdot}124 \text{ (approx.)}$$

∴ the height of the chimney is 19·124 metres (approx.).

Example 6: The angle of elevation of the top of a monument when observed from a point on the roof of a five-storied building, 18 m high, is 45° and the angle of depression of the foot of the monument, when observed from the same point, is 60°. What is the height of the monument?



Given that,

Fig. 38

AB=height of the five-storied building=18 m, $\angle CAP = 45°$, $\angle PAD = 60° = \angle ADB$. Let the height of the monument= $CD = x$ m $\therefore$ $CP=(x-18)$ m.

Also let $BD=y$ m$=AP$

Then, tan $\angle CAP$=tan 45°

$$\therefore \frac{CP}{AP}=\frac{x-18}{y}=1 \qquad \ldots(i)$$

Again, tan $\angle ADB$=tan 60°

$$\therefore \frac{AB}{BD}=\frac{18}{y}=\sqrt{3} \qquad \ldots(ii)$$

Dividing equation no. *(i)* by equation no. *(ii)*, we get,

$$\frac{x-18}{18}=\frac{1}{\sqrt{3}}$$

or, $x\sqrt{3}-18\sqrt{3}=18$

or, $x\sqrt{3}=18\sqrt{3}+18$

$$\text{or, } x = \frac{18(\sqrt{3}+1)}{\sqrt{3}} = 6\sqrt{3}(\sqrt{3}+1)$$
$$= 6(3+\sqrt{3}) = 6 \times 4{\cdot}732 \text{ (approx.)}$$
$$= 28{\cdot}392 \text{ (approx.)}$$

∴ the height of the monument is 28·392 metres (approx.).

Example 7: A ship lies at anchor at a distance from the sea shore. From a point on the deck of the ship the angles of elevation of the top and foot of a light-house standing on the sea shore are 60° and 30° respectively. If the light-house is at a height of 8 metres from the sea level and the deck of the ship is at a height of 3 metres from the sea level, then what is the height of the light-house?

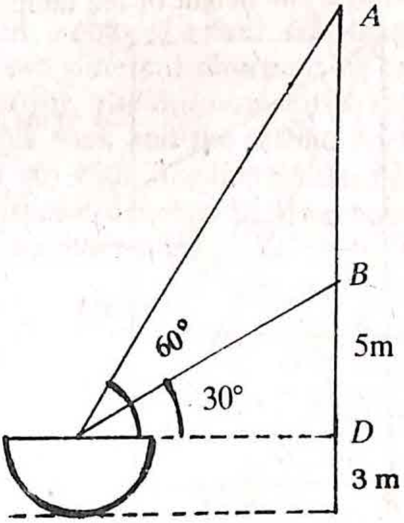


Fig. 39

Let the height of the light-house $= AB = x$ metres.

$\angle BOD = 30°$, and
$\angle AOD = 60°; BC = 8$ metres and $DC = 3$ metres.
$\therefore BD = 5$ metres.

$$\therefore \frac{BD}{OD} = \frac{5}{y} = \tan 30° = \frac{1}{\sqrt{3}} \qquad \ldots(i)$$

Again, $\frac{AD}{OD} = \frac{x+5}{y} = \tan 60° = \sqrt{3}$. (ii)

Dividing equation no. (ii) by equation no. (i), we get,

$$\frac{x+5}{5} = 3$$

or, $x+5 = 15$

or, $x = 15-5$

or, $x = 10$

∴ the height of the light-house is 10 metres.

Example 8: From a point on the roof of a house, 11 metres high, it is observed that the angles of depression of the top and foot of a lamp post are 30° and 60° respectively. What is the height of the lamp post?

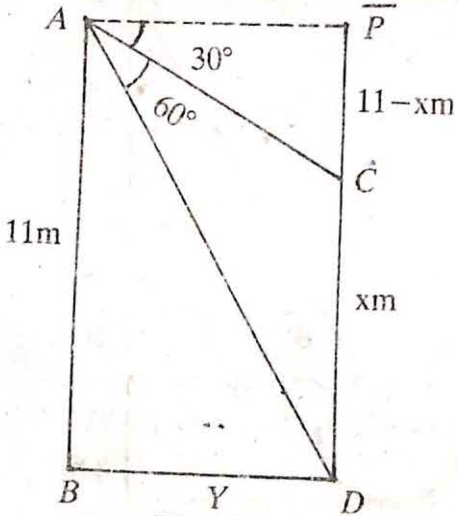


Fig. 40

Given that, the height of the house $AB=11$ m, $\angle PAC = 30°$ and $\angle PAD = 60°$; and $PD=AB=11$ m. Let the height of the lamp post $CD=x$ m.

∴ $PC=(11-x)$ m.

Also, let $AP=y$ m.

Now, $\tan \angle PAC = \frac{PC}{AP} = \tan 30°$.

$$\therefore \frac{11-x}{y} = \frac{1}{\sqrt{3}} \quad \text{...(i)}$$

Again, $\tan \angle PAD = \dfrac{PD}{AP} = \tan 60°$

$\therefore \dfrac{11}{y} = \sqrt{3}$ ...(ii)

Dividing equation no. (i) by equation no. (ii), we get,

$\therefore \dfrac{11-x}{11} = \dfrac{1}{3}$

or, $33-3x = 11$

or, $3x = 33-11$

or, $x = \dfrac{22}{3} = 7\frac{1}{3}$

$\therefore$ the height of the lamp post is $7\frac{1}{3}$ metres.

Example 9: From a quay of a river, 600 metres wide, two boats start in two different directions to reach the opposite side of the river. The first boat moves making an angle of 30° with this bank and the second boat moves making an angle of 90° with direction of the first boat. What will be the distance between the two boats when both of them reach the other side?

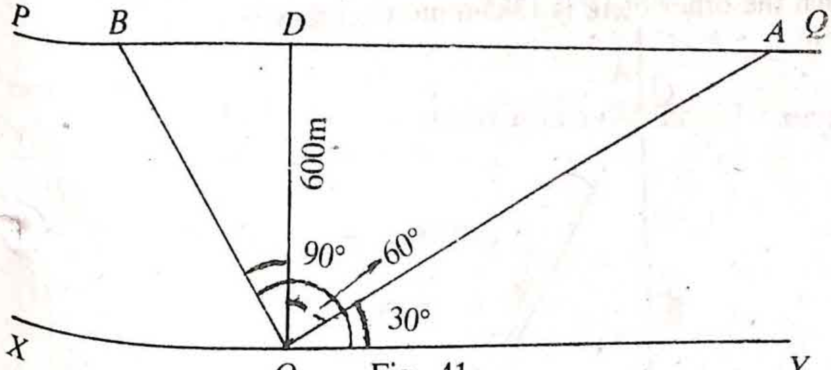


Fig. 41

Let XY be one bank of the river and the boats start from a point O on this bank. Let OA be the direction of the first boat and OB be that of the second boat. A, B are the respective points, where the first and second boats reach the opposite bank.

To find the distance AB.

Given that, $\angle AOY = 30^\circ$, $\angle AOB = 90^\circ$ and $OD = 600$ m.

$\therefore \angle DOA = 60^\circ$ and $\angle BOD = 30^\circ$

Let $AD = x$ metres and $BD = y$ metres.

Now we get from right-angled triangles $\triangle BOD$ and $\triangle AOD$,

$$\frac{x}{600} = \tan 60^\circ = \sqrt{3} \qquad \ldots(1)$$

$$\text{and } \frac{y}{600} = \tan 30^\circ = \frac{1}{\sqrt{3}} \qquad \ldots(2)$$

$\therefore$ adding equation nos. (1) and (2) we get,

$$\frac{x+y}{600} = \sqrt{3} + \frac{1}{\sqrt{3}}$$

$$\text{or, } \frac{x+y}{600} = \frac{3+1}{\sqrt{3}} = \frac{4}{\sqrt{3}}$$

$$\text{or, } x+y = \frac{4}{\sqrt{3}} \times 600 = 4 \times 200\sqrt{3} = 800\sqrt{3}$$

$$= 800 \times 1{\cdot}732 \text{ (approx.)} = 1385{\cdot}6 \text{ (approx.)}$$

$\therefore$ The distance between the two boats when they reach the other bank is 1385·6 metres (approx.).

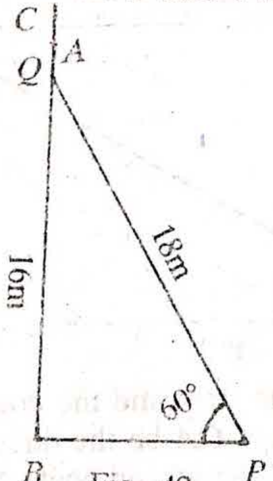


Fig. 42

Example 10: A ladder is kept in contact with one wall of a five-storied building, vertically on the ground and it is found that the top of the ladder is 2 metres high from the roof of the building. The foot of the ladder is moved away perpenducularly from the bottom edge of that wall and it makes an angle of 60° with the line that joins the foot of the ladder and the point at the bottom of the building just below the point with which the ladder was in contact. If the height of the building is 16 metres, what is the distance of the foot of the ladder from the wall? How below is the top of the ladder from the roof at present?

From adjoining figure, AB = height of the five-storied building = 16 metres,

PQ = length of the ladder

$BC = (16 + 2)$ metres

$= 18$ metres

and $\angle BPQ = 60°$

To find the lengths of PB and AQ,

(i) Let BP = x metres,

$$\therefore \frac{BP}{PQ} = \frac{x}{18} = \cos 60° = \frac{1}{2}$$

$$\therefore x = \frac{18}{2} = 9$$

$\therefore$ The foot of the ladder is at a distance of 9 metres from the wall at present.

(ii) Now, let BQ = y metres,

$$\therefore \frac{BQ}{BP} = \frac{y}{9} = \tan 60° = \sqrt{3}$$

$\therefore y = 9\sqrt{3} = 9 \times 1{\cdot}732$ (approx.) $= 15{\cdot}59$ (approx.)

$\therefore$ The top of the ladder is below the roof by a distance of $(16 - 15{\cdot}59)$ metres (approx.), that is, $0{\cdot}41$ metre, (approx.)

or, $\frac{9}{x-y}=\frac{1}{\sqrt{3}}$ (ii)

$\therefore x-y = 9\sqrt{3} = 9 \times 1{\cdot}732$ (approx.) $= 15{\cdot}588$ (approx.)

Substituting the value of x obtained from equation no. (*i*), we get,

$34{\cdot}615 - y = 15{\cdot}588$
or, $y = 34{\cdot}615 - 15{\cdot}588$
$= 19{\cdot}027$ (approx.).

That is, the boat traverses approximately 19·027 metres in 40 seconds.

$\therefore$ Speed of the boat

$= \frac{19{\cdot}027}{40} \times 60 \times 60$ metres per hour (approx.)

$= 1712{\cdot}43$ metres per hour (approx.).

Therefore (*i*) The width of the canal = 34·615 m. (approx.)
(*ii*) The speed of the boat = 1712·43 m/hr. (approx.)

Exercise 5

1. The angle of elevation of the top of a coconut tree from a point on the ground is 60° If the point is 20 m apart from the foot of the tree, what is the height of the tree?

2. One flies a kite with a thread 150 m long. If the thread of the kite makes an angle of 60° with the horizontal line, how high is the kite from the ground?

3. Fire breaks at the second floor of a house. Fire brigadiers fix the top of a ladder at one window of the second floor and throw water by a hose pipe. If the ladder is $6\sqrt{3}$ metres long and makes an angle of 60° with the ground, find the height where the window stands.

4. A telegraph post is bent at a point above the ground due to storm. Its top just meets the ground at a distance of $8\sqrt{3}$ metres from its foot and makes an angle of 30°. At what height is the post bent? What is the height of the post?

5. Two houses stand just on the opposite sides of a road. A ladder that stands against the wall of the first house is at a distanc of 6 metres from the house and makes an angle of 30° with the horizontal line: But if the ladder stands against the wall of the second house keeping its foot at the same point, it makes an angle of 60° with the horizontal line. Assuming that the points where the top of the ladder is fixed are just above the opposite sides of the road. Find

(i) the length of the ladder,

(ii) the distance of the foot of the ladder from the wall of the second house,

(iii) the width of the road,

(iv) the height where the top of the ladder is fixed with the wall of the second house.

6. The angle of elevation of the top of a chimney from a point on the horizontal line through the foot of the chimney is 60°. The angle of elevation from another point on the same line at a distance of 24 metres from the first point is 30°. What is the height of the chimney?

[Find the approximate value correct to three decimal places, assuming $\sqrt{3} = 1{\cdot}732$ (approx.)].

7. A palm tree stands on the bank of a river. A post is fixed in the earth on the other bank just opposite to the palm tree. On moving $7\sqrt{3}$ metres from the post along the bank it is found that the tree makes an angle of 60° at that point with respect to this bank. Find the width of the river.

8. The length of the shadow of a tower is 9 metres when sun's altitude is 30°. What is the height of the tower?

9. When sun's altitude increases from 45° to 60°, the shadow of a post is diminished by 3 metres. What is the height of the post? [Find approximately correct to three decimal places taking $\sqrt{3}=1{\cdot}732$ (approx.)].

10. When the top of a chimney of a factory is seen from a point, on the roof of a three-storied building, just opposite to the chimney, the angle of elevation is 30°. If the building is $9\sqrt{3}$ metres high and the distance between the factory and the chimney is 30 metres, what is the height of the chimney?

11. Three points one each at the bottom of two ships and a light-house are on the same straight line. From a point at the top of the light-house just vertically above the point at the bottom, the angles of depression of the points at the bottom of the ships are 60° and 30° respectively. If the distance of the points at the bottom of the light-house and at the bottom of the first ship is 150 metres, what is the distance between the two points, one at the bottom of the light-house and the other at the bottom of the other ship? What is the height of the light-house?

12. From a point on the roof of a five-storied building the angle of elevation of the top of a monument and that of depression of the foot of the monument are 60° and 30° respectively. The height of the building is 16 metres. Assuming that the distance of the point on the roof from the monument is same as the distance between the monument and the building, find the height of the monument. How far is the building from the monument?

13. The thread of a flying kite is 250 metres long. When the thread makes an angle of 60° with the horizontal line through the end of the thread fixed with the reel, what is the height of the kite from that point? If the thread makes an angle of 45°, what is the height? In which of these two cases, will the kite be at a greater height from the ground?

14. A passenger in an aeroplane observes that Howrah Station is on one side of the plane and Saheed Minar is just on the opposite side. The angles of depression of Howrah Station and Saheed Minar (which are taken to be two points) as observed by that passanger are 60° and 30° respectively. If the aeroplane is at a height of $545\sqrt{3}$ metres at that time, what is the distance between Howrah Station and Saheed Minar?

15. The length of the flag post set at the roof of the second floor of the Writers' Building is 3·3 metres. From a point of B.B.D. Bag the angles of elevation of the top and foot of the post are 50° and 45° respectively. Find the height of the Writers' Building in metres approximately correct to two decimal places. (Take $\tan 50^\circ = 1{\cdot}192$)

16. The heights of two towers are 180 metres and 60 metres respectively. If the angle of elevation of the top of the first tower from the foot of the second tower is 60°, what is the angle of elevation of the top of the second tower from the foot of the first?

17. The shadow of a tower becomes 60 metres longer when the altitude of the sun changes from 45° to 30° What is the height of the tower?

18. From a point on the horizontal line through the foot of a chimney the angle of elevation of the top of the chimney is 30° and the angle of elevation of the top of the chimney is 60° at a point on the same straight line 50 metres nearer to the chimney. What is the height of the chimney?

19. A boat is approaching straightaway a bridge on a canal. From a point on the board of the boat the angle of elevation of a point on the bridge just vertically above from the straight line in which the boat is moving is 30°. The boat reaches below the bridge proceeding 24 metres along the same straight path. What is the height of the bridge from the board of the boat?

20. A vertical post, 126 dcm high is bent at a point above the ground and it just meets the ground making an angle of 30° with the ground. At what height was the post bent and at what distance does it meet the ground from the foot of the post?

21. The angle of depression of the top of a fountain from a point of the window on the fourth floor of a multi-storied building is 45°; but the angle of elevation of a point of the window on the first floor vertically below the point on the fourth floor from the top of the fountain is 30°. The distance of these two points from the fountain are the same as distance between the building and the fountain, which is 12 metres. What is the distance between the two windows? ($\sqrt{3} = 1{\cdot}732$)

22. A man standing in the midst of a field observes a flying bird in his north at an angle of elevation of 30° and after 2 minutes he observes the bird in his south at an angle of elevation of 60°. If the bird flies in a straight line all along at a height of $50\sqrt{3}$ metres, what is its speed?

23. A man standing on a railway overbridge $5\sqrt{3}$ metres high observes the top-most point of the front of the engine at an angle of depression of 30°. But just after 2 seconds he observes the same point on the engine at an angle of depression of 45°. The point vertically below the point where the man stands are in the same straight line with two positions of the point on the engine. Find the speed of the train. (Find in kilometres per hour approximately correct to three places of decimals, assuming $\sqrt{3} = 1{\cdot}732$ (approx.).

24. Two boats start from Bhatpara quay making an angle of 75° with each other and reach at two separate quays on the other side of the river. The first boat moves making an angle of 60° with this side of the river. Both the boats take 10 minutes to reach the other side. If width of the ganges at that time is 1200 metres, then *(i)* What was the speed of the first boat? *(ii)* What was the distance between the two quays on the other side? *(iii)* What was the speed of the second boat? ($\sqrt{3} = 1{\cdot}732$)

25. In the morning Anupam standing in the midst of a field observes a bird to fly from east to west. When the bird is just over his head he observes that its shadow falls on the ground at a distance of 200 metres from him. If the altitude of the sun is 30°, what was the height of the bird from the ground?

26. A betel-nut tree is bent at a height of 15 metres above the ground and its top just meets the ground. If this bent part makes an angle of 60° with the ground, what was the height of the tree and at what distance does its top meet the ground from the foot of the tree?

27. Two light post of equal height stand just on opposite sides of a road, 12 metres wide. A multi-storied building is on the same vertical plane with them. Murshid observes from a point on the roof of the building the tops of the light posts at angles of depression of 60° and 45° respectively. If the point at the bottom of the building just vertically below the point from where Murshid observes and the foots of light posts are on the same straight line, find the distance of the house from the road. If each light post is 5 metres high, what is the height of the building?

28. A flag post stands on a monument. When sun's altitude is 30° the length of the shadow of the flag post is $3\sqrt{3}$ metres. What was the length of the flag post? If the height of the monument is 15 metres, what will be the length of the shadow of the monument together with that of the flag post?

29. A bridge stands at right angles to the bank of the river. If one moves away a certain distance from the bridge along this side of the river the other end of the bridge is seen at an angle of 45°. If that person moves a further distance of 400 metres in the same direction the other end is seen at an angle of 30°. What is the length of the bridge?

30. A house, 15 metres high, stands on one side of a park and from a point on the roof of the house the angle of depression of the foot of a chimney is 30° and the angle of elevation of the top of the chimney is 60°. The distance of the point on the roof from the chimney is the same as the distance between the house and the chimney. What is the height of the chimney? What is the distance between the house and the chimney?

31. Babai observes, from a point on the roof of the house, Papai at a point on the window on the ground floor of a five-storied building which is just opposite on the other side of a field and also observes Tatai at a point on the window on the fourth floor of the same building. The distance of these two points from the vertical line through the point from where Babai observes is the same as the distance between the house of Babai and the five-storied building. The angle of depression at which Babai observes Papai is 30°. and the angle of elevation at which Babai observes Tatai is also 30°. If the distance between two houses is 24 metres and the point on the window of the ground floor is at a height of $2\sqrt{3}$ metres, find the distance between Papai and Tatai and also the height of the house of Babai.

32. Bablu was flying a kite in the morning with a long thread. The kite was on the western sky making an angle of 60° with the ground. The shadow of the kite falls on the ground at a distance of $\frac{200}{\sqrt{3}}$ metres from Bablu. If the altitude of the sun is 60°, find *(i)* the height of the kite from the ground, *(ii)* the length of the thread that Bablu released. (Ignore the height of the reel from the ground and take $\sqrt{3} = 1{\cdot}732$.)

Geometry

Exercise 1

1. 55°, equiangular **2.** *(i)* 16 rt. angle, 4 rt. angles, 20 rt. angles, 4 rt. angles *(ii)* 9 *(iii)* 12 *(iv)* 140°, 40° **3.** 64° **4.** 55°, 117·5°, 55°, 117·5°, 70° **5.** *(i)* non-collinear *(ii)* collinear *(iii)* collinear *(iv)* non-collinear *(v)* collinear *(vi)* collinear *(vii)* collinear *(viii)* collinear **6.** 90° **7.** 90° **8.** $AB=AD$ **9.** 135°, 45°, 135° **10** 60°, OA, OD; OB; OE; OC; OF; regular hexagon **11.** *(i)* BC, AB, AC *(ii)* $\angle A$, $\angle C$, $\angle B$ **12.** $\angle AOD=108°$, $\angle DOB=72°$, $\angle BOC=108°$, $\angle COA=72°$ *(i)* $\angle BAC + \angle ACD$ = 2 rt angles *(ii)* $\angle ABC = \angle BCD$ *(iii)* (A) $\angle CDB+\angle ABD$=2 rt. angles (B) $\angle CDB=\angle ABE$ **14.** *(i)* and *(ii)* possible, in case of *(i)* sides 2 cm and 6 cm are on the same side of the diagonal and sides 5 cm and 4 cm are on the other side of the diagonal, in case of *(iii)* sides may be taken in any order. **15.** 12 cm. **16** OR, smallest, $OQ=OS$, no **17** *(i)* If two adjacent angles are equal and if two exterior angles are in a straight line, then each angle will be a right angle. *(ii)* Of two triangles sides and the included angle of one is equal to two sides and the included angle of the other, then those two triangles are congruent. *(iii)* Of two triangles if two angles and one side of one is equal to two angles and the corresponding side of the other, then those two triangles are congruent. *(iv)* If two triangles are congruent then the corresponding angles and corresponding sides will be equal. *(v)* *(a)* Since two corresponding angles $\angle BGE$ and $\angle DHG$ are equal, therefore, AB and CD are parallel *(b)* Since two alternate angles $\angle BGH$ and $\angle GHC$ are equal, therefore, AB and CD are parallel. **18** *(i)* $\angle A=\angle Q$, $\angle C=\angle P$ *(ii)* $PR=XY$, $PQ=YZ$ *(iii)* $\angle A=\angle Z$, $\angle B=\angle Y$, $\angle C=\angle X$

Exercise 2

1. parallelogram **2.** $ABEC$, $ABFD$ **3.** $AC \parallel DB$, $AD \parallel CB$ **4.** parallelogram **5.** $AB \parallel EF$, rectangle **6.** $AD \parallel BC$ **7.** concurrent **8.** each is a parallelogram. *(i)* $AP=AQ$ *(ii)* $BP=BR$ *(iii)* $CR=CQ$ *(iv)* $AB=\frac{1}{2}\,QR$ *(v)* $BC=\frac{1}{2}\,PQ$ *(vi)* $CA=\frac{1}{2}\,RP$ **9.** perpendicular bisectors are concurrent **10.** internal bisectors are concurrent **11.** medians are concurrent **12.** parallelogram.

Exercise 3

1. 360°; 180°; $AB \parallel DC$ **8.** $BCEF$ would be a parallelogram.

Exercise 24

1. *(i)* 6 units *(ii)* 9 units *(iii)* 4 units **2.** *(i)* No *(ii)* Yes.

Exercise 28

2. On the circle **3.** The point D lies on the circle **4.** Radii are equal. **5.** The point B lies on that circle, $ABCD$ is an isosceles trapezium **6.** The point of intersection of the diagonals will be the centre. the radius will be half of any of the diagonals.

Exercise 29

1. *(i)* 1·6 cm (approx.) *(ii)* 1·9 cm (approx.) *(iii)* 2·2 cm (approx.) *(iv)* 2·3 cm (approx.) *(v)* 2 cm (approx.) *(vi)* 2·1 cm (approx.) **2.** 4 cm (approx.) 2 cm (approx.) Circum-radius is twice the in-radius.

Exercise 30

4. Parallel **5.** Square

Exercise 31

1. 6 cm **2.** At the point B or C **3.** Common tangent **4.** 90°

Exercise 33

2. 10 cm, 8 cm.

Exercise 34

1. *(i)* 3 cm *(ii)* 3·5 cm (approx.) *(iii)* 5·7 cm (approx.) *(iv)* 5·7 cm (approx.) *(v)* 5·5 cm (approx.) *(vi)* 6 cm **3.** *(i)* 2·6 (approx.) *(ii)* 3·9 (approx.) *(iii)* 3·5 (approx.) *(iv)* 4·2 (approx.) *(v)* 4·9 (approx.) *(vi)* 5·5 (approx.)

Miscellaneous Exercise 7

1. 4 cm, will pass **2.** 7 cm **3.** 6·5 cm **4.** 2 cm **5.** 13 cm **6** 10 cm **7.**

(ii) **8.** 91° **9.** 70° **10** 114° **11.** 90°, 65° **12.** $\angle BDC=125^\circ$, $\angle DCA=125^\circ$, $\angle CAB=55^\circ$, $\angle ABD=55^\circ$ **13.** $\angle BAC=60^\circ$, $\angle ABC=50^\circ$, $\angle BCA=70^\circ$; $\angle EAC=20^\circ$, $\angle CEA=130^\circ$, $\angle ACE=30^\circ$ **14.** 8 cm, 8 cm, 9·6 cm, 3·6 cm. **15.** 12 cm, 6 cm **16.** 5 cm, **17.** 5 cm, 4 cm, 3 cm; right-angled triangle.

Miscellaneous Exercise 8

1. *(i)* three *(ii)* bisects *(iii)* 90° *(iv)* half *(v)* equal *(vi)* right angle, acute angle, obtuse angle *(vii)* supplementary *(viii)* concyclic *(ix)* cyclic *(x)* perpendicular *(xi)* equal, equal, bisector, bisects, at, right, angles *(xii)* straight, line, equal, sum, radii (xiii) collinear, equal, difference, radii. *(xiv)* proportionally *(xv)* parallel *(xvi)* corresponding, proportional *(xvii)* proportional, equiangular (or similar) *(xviii)* similar, themselves, similar *(xix)* hypotenuse, equal, sum, squares *(xx)* a right.

2. *(i)* bisects *(ii)* concurrent, point, on *(iii)* isosceles *(iv)* rectangle *(v)* diameter *(vi)* parallel *(vii)* perpendicular, bisector *(viii)* equal *(ix)* point. intersection, perpendicular, bisectors *(x)* bisector.

3. *(i)* Only one circle can be drawn through any three non-collinear points. *(ii)* Correct *(iii)* An angle at the centre of a circle is double the angle at the circumference standing on the same arc. *(iv)* The corresponding sides of equiangular triangles are proportional. *(v)* The perpendicular drawn to the end point (which is not the centre) of a radius of a circle is a tangent to the circle at that point. *(vi)* Tangents to a circle at any two points (which are not the end points of a diameter) on the circle must intersect each other. *(vii)* Correct *(viii)* Correct *(ix)* Correct *(x)* Correct.

Miscellaneous Exercise 9

1. Chord, bisecting any two sides at right angle, the line segment that joins the point of intersection of two perpendicular bisectors and any of the vertices **2.** Tangent, bisecting any two angles, drawing perpendicular to any side from the point of intersection of the angle-bisectors **3.** They are situated at right angles **4.** 90°, 90° **5.** *(i)* right-angled triangle *(ii)* equal to the difference of radii of two circles *(iii)* drawing two arcs— one with the mid-point of AB as centre and half of AB as radius, and the other with centre A and the difference of the radii of two circles as radius (*or*, drawing a circle with centre A and the difference of the radii of two circles as radius, and drawing a semi-circle with AB as diameter) *(iv)* P is the point where AR produced meets the circle *(v)* Drawing parallel to RB through the point P **7.** Rectangle **8.** *(i)* Right-angled triangle *(ii)* equal to the sum of the radii of two circles *(iii)* drawing two arcs— one with the mid-point of AB as centre and half of AB as radius, and the other with A as centre and the sum of the radii of two circles as radius (*or*, drawing a circle with centre A and the sum of the radii of two circles as radius and drawing a semi-circle with AB as diameter) *(iv)* P is the point where AR meets the circle *(v)* Drawing parallel to RB through the point P **10.** $b^2=ac$, AD, AD, $BC=a+b$, will have to draw a semi-circle with BC as diameter, drawing perpendicular to BC from the point D. **11.** AC, $BD=a-b$, will have to draw a semi-circle with BC as diameter **12.** Right-angled triangle, $\sqrt{ab}$ is the mean-proportional of a and b **13.** 4·9 (approx.)

Miscellaneous Exercise 10

27. No. The property 'If two triangles are equiangular, their corresponding sides are proportional' that has been used to prove the theorem, but this property has again been used to prove the above mentioned property.

Trigonometry

Exercise 1

1. *(i)* 13°52′ *(ii)* 2°1′52″ *(iii)* 6′15″ *(iv)* 27°5′ *(v)* 72°2′24″ **2.** *(i)* $\frac{\pi}{4}$ *(ii)* $\frac{3\pi}{4}$ *(iii)* $-\frac{5\pi}{6}$ *(iv)* $\frac{3\pi}{20}$ *(v)* $\frac{\pi}{8}$ *(vi)* $-\frac{3\pi}{8}$ *(vii)* $\frac{47\pi}{160}$ **3.** 100°, $\frac{5\pi}{9}$ **4.** $\frac{\pi}{4}$ **5.** 75°, 60° and $\frac{5\pi}{12}, \frac{\pi}{3}$ **6.** Two complete revolutions and an angle of 195°; clockwise direction **7.** 87°, $\frac{29\pi}{60}$ **8.** 60° **9.** $\angle BAD=\angle BCD=\frac{3\pi}{8}$; $\angle ABD=\angle CBD=\frac{\pi}{6}$ **10.** $\angle BAD=\angle DBC=\frac{3\pi}{8}$, $\angle ABD=\angle BCD=\frac{\pi}{8}$ and $\angle ADB=\angle BDC=\frac{\pi}{2}$ **11.** $\angle EAC=\angle AEC=\frac{\pi}{6}$; $\angle ACE=\frac{2\pi}{3}$.

Exercise 2

1. $\frac{3}{5}, \frac{3}{4}$ **2.** $\frac{15}{17}, \frac{17}{8}$ **3.** $\frac{8}{17}, \frac{17}{15}$ **4.** $\frac{3}{4}, \frac{5}{3}$ **5.** $\frac{12}{13}, \frac{5}{13}, \frac{12}{5}$ **6.** $\frac{5}{7}$ **8.** *(i)* $\frac{1}{2}$ *(ii)* $\sqrt{2}+1$ *(iii)* 0 *(iv)* 0 *(v)* $\frac{17}{13}$ *(vi)* 0 *(vii)* $\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ *(viii)* $\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ *(ix)* $\frac{4}{3}$ *(x)* $\frac{1}{2}$ *(xi)* $\pm\frac{3}{2}$ *(xii)* 1 *(xiii)* $\frac{4}{\sqrt{3}}, \pm\frac{2}{\sqrt{3}}$ $\sqrt{3}$ or $\frac{1}{\sqrt{3}}$ *(xiv)* $\frac{13}{12}$ *(xv)* $\frac{1}{\sqrt{5}}$ *(xvi)* $\frac{1}{\sqrt{6}}$ **10.** $\frac{n}{\sqrt{m^2+n^2}}, \frac{\sqrt{m^2+n^2}}{n}$

Exercise 3

1. 0·6, 0·8, 0·75, 1·67, 1·25, 1·33 and 0·8, 0·6, 1·33, 1·25, 1·67, 0·75 **3.** *(i)* $\frac{1}{2}$ *(ii)* 0 *(iii)* 1 *(iv)* $3\frac{1}{3}$ *(v)* $\frac{6+\sqrt{6}}{3}$ *(vi)* 0 *(vii)* 9 *(viii)* $\frac{5}{2\sqrt{3}}$ *(ix)* 0 *(x)* $\frac{3\sqrt{3}+1}{2\sqrt{3}}$ *(xi)* $5\frac{1}{3}$ *(xii)* 1 **6.** $\frac{1}{\sqrt{3}}$ **7.** $2\frac{2}{3}$ **8.** $\pm\frac{1}{2}$ **9.** 530 m.

Exercise 5

1. $20\sqrt{3}$ m **2.** $75\sqrt{3}$ m **3.** 9 m **4.** 8 m, 24 m **5.** *(i)* $4\sqrt{3}$ m *(ii)* $2\sqrt{3}$ m, *(iii)* $2(3+\sqrt{3})$ m *(iv)* 6 m **6.** 20·784 m **7.** 21 m **8.** $3\sqrt{3}$ m **9.** 7·098 m. **10.** $19\sqrt{3}$ m **11.** 450 m, $150\sqrt{3}$ m **12.** 64 m, $16\sqrt{3}$ m **13.** $125\sqrt{3}$ m, $125\sqrt{2}$, in the first case **14.** 2180 m **15.** 17·19 m (approx.) **16.** 30° **17.** 81·96 m (approx.) **18.** $25\sqrt{3}$ m **19.** $8\sqrt{3}$ m **20.** 42 dcm, $42\sqrt{3}$ dcm **21.** 5·072 m. **22.** 6 km/hr. **23.** 42·588 km/hr. (approx.) **24.** *(i)* 8·32 km/hr. (approx.), 1892·8 m, 10·18 km/hr, (approx.) **25.** 115·47 m (approx.) **26.** 32·32 m, 8·66 m (approx.) **27.** 16·392 m, 33·392 m **28.** 3 m, $18\sqrt{3}$ m **29.** $200(\sqrt{3}+1)$ m **30.** 45 m, $15\sqrt{3}$ m., **31.** $16\sqrt{3}$ m, $10\sqrt{3}$ m, **32.** 100 m 115·47 m (approx.)

Mensuration

Miscellaneous Exercise

1. 50 sq. m. and 50 m. **2.** 864 sq. m. **3.** 4200 sq. m. **4.** 64 sq. m. **5.** Rs. 162 **6.** 2000 tiles **7.** 659·34 sq. m., 62·64 sq. m. **8.** 35 sq. cm. **9.** 45 cm., **10.** height=$4\sqrt{3}$ cm, area = $16\sqrt{3}$ sq. m. **11.** perimetre = $30\sqrt{3}$ cm, area = $75\sqrt{3}$ sq. cm. **12.** 42 sq. cm.

13. 4·615 cm (approx.) **14.** 24 cm. **15.** 12 sq. cm. **16.** 12 cm., 48 sq. cm. **17.** 15 cm. **18.** 48 cm, 109 sq. cm. **19.** 12 cm, **20.** $40\sqrt{3}$ sq. cm. **21.** area = $108\sqrt{15}$ sq. m, $\frac{9\sqrt{15}}{2}$ m. **22.** 3·5 km, Rs. 33·075 **23.** 20 m. **24.** perimetre = 24 dcm, area = 24 sq. dcm. **25.** 15 m. **26.** $324\sqrt{6}$ sq. m. **27.** $30\sqrt{13}$ km. **28.** 26 m. **29.** 16 : 9 **30.** 15 cm.

Exercise 2

1. 132 m. **2.** 110 cm. **3.** Rs. 114·40 **4.** 12500 **5.** 2, 5 **6.** 17·5 cm. **7.** 1540 m, 154 m. **8.** 46·2 km. **9.** 7 cm. **10.** 4·4 dcm. **11.** 21 cm. **12.** 8 cm, 12 cm. **13.** 63 cm. **14.** 3·5 cm. **15.** 77m, 21 m. **16.** 22 cm. **17.** 14400

Exercise 3

1. *(i)* 5544 sq. m, *(ii)* 154 sq. m, *(iii)* 124·74 sq. m, *(iv)* 75 sq, m. 46 sq. dcm. **2.** 154 sq. cm. **3.** 42 m, 1386 sq. m. **4.** 28 m. **5.** 16·94 sq. m. **6.** 1232 sq. m. **7.** 550 sq. cm. **8.** 56 cm, 196 sq. cm. **9.** 42 sq. m. **10.** perimetre of square: circumference = $2\sqrt{7}:\sqrt{22}$ **11.** 21 cm. **12.** 1389850 sq. m. **13.** 45·5 sq. cm. **14.** 346·5 sq. m. **15.** 6 kg 160 gr, 154 gr. **16.** 9:25:49 **17.** 22176 sq. m.

Miscellaneous Exercise

1. $A=50$ sq. cm; $A=\frac{25\sqrt{3}}{4}$ sq. cm, $h=\frac{5\sqrt{3}}{2}$ cm; $A=36$ sq. cm; $l=b=2\sqrt{5}$ cm; $l=b=9\sqrt{2}$ m, $A=162$ sq. m; $A=32$ sq. m; $A=2\sqrt{21}$ sq. m, $h=\sqrt{21}$ m; $A=9{\cdot}922$ sq. cm (approx.); $A=6$ sq. m; $A=154$ sq. cm; $A=38{\cdot}5$ sq. m; A (of circle)$=14\frac{1}{7}$ sq. m **2.** *(i)* 39 cm *(ii)* 27 cm *(iii)* 84 dcm *(iv)* 36 m *(v)* 12 m *(vi)* 28 cm *(vii)* $64\sqrt{2}$ m *(viii)* 20 cm *(ix)* 26 cm *(x)* 44 mm **3.** *(i)* 10 m *(ii)* 30 m *(iii)* $4\sqrt{3}$ m, $16\sqrt{3}$ sq.m *(iv)* $12\sqrt{3}$ cm, $108\sqrt{3}$ sq. cm **4.** 30 sq. cm **5.** 5·4 m **6.** 12·5 m **7.** $12\sqrt{7}$ sq. cm **8.** 108 m, 144m, 180 m, 432 m **9.** $16\sqrt{3}$ sq. cm **10.** 2 km apart from each village **11.** 459 sq. m **12.** $AD=234$ m, $BC=270$ m **13.** 8 cm **14.** 6400 pieces **15.** 484 sq. m, 212·5 sq. m **16.** 30 cm **17.** Rs. 1075 **18.** 3023·75 sq. m, Rs. 3045·00 **19.** 437·5 sq. m, 587·5 sq. m **20.** 346·5 sq. cm **21.** 14 min. 24 sec. **22.** Rs. 1559·25 **23.** 14 cm, 21 cm; 16 : 5 : 4.

Exercise 4

1. Rs. 100·40 **2.** Rs, 374·00 **3.** Rs. 121·28 **4.** Rs. 201·00 **5.** 3 m **6.** 5·5 m, 4·4 m, 3·33 m. **7.** 2·04 cu. m. **8.** 3500 cu. m., Rs. 14875·00 **9.** 11 pieces **10.** 850 kilolitres **11.** 1·1 m **12.** 150 m **13.** 200 gms **14.** 13·75 cu. m **15.** 135 pieces **16.** 0·75 m **17.** 18 litres **18.** Length 2·2 m, breadth 1·6 m **19.** 26·25 c.c, 1·5 cm **20.** 10000 pieces **21.** 2 dcm **22.** 2 m, 1·5 m; No, because only 1500 litres of oil can be contained in the tub **23.** 1 m, 8 dcm **24.** 2040 pieces **25.** 6880 litres **26.** 2 metres **27.** 5 dcm, Rs. 258·50 **28.** 7 hrs. 24 min, 5 dcm.

Exercise 5

1. Rs. 145·53 **2.** Rs. 201·96 **3.** Rs. 554·40 **4.** 6 dcm **5.** 6·4 dcm **6.** Rs. 277·20 **7.** 1386 c.c. **8.** 3326·4 cu. dcm **9.** 14437·5 litres **10.** 27·72 cu. m **11.** 1232 cu. dcm, 704 cu. dcm, 352 cu. dcm, 176 cu. dcm. **12.** 1232 c.c. **13.** 16 pieces **14.** 1·5 kg **15.** 325 gms **16.** 13365 litres **17.** 7700 litres **18.** 7 cm **19.** 2660 litres **20.** 7 hrs. **21.** 3·2 cm **22.** 4·05 dcm. **23.** 1·4 m **24.** $\frac{2}{9}$ parts **25.** 13 min. **26.** 2·1 m **27.** 200 metres **28.** 3·08 cu. m, 0·84 cu. m. **29.** 302·4 cu. dcm, 172·8 cu. dcm. **30.** 13·86 cu. dcm **31.** 134·75 cu. m, 19·25 cu. m **32.** 41·58 cu. dcm, 2 quintals 7·9 kg. **33.** 277·2 cu. dcm, 55·44 cu. dcm **34.** *(i)* 8·25 m *(ii)* 9240 litres *(iii)* 18·48 litres **35.** *(i)* $\frac{4}{5}$ *(ii)* 27720 litres *(iii)* 55440 litres.

Exercise 6

1. 264 sq. cm **2.** Rs. 132 **3.** 24 cm **4.** 38·5 sq. m **5.** $427\frac{3}{7}$ sq. m, 17 m **6.** 0·616 cu. dcm **7.** 1617 cu. m **8.** 23·1 sq. m, 12·936 cu. m **9.** $37\frac{5}{7}$ cu. m, 4 m **10.** $4\frac{5}{7}$ c.c. **11.** 2·4 km **12.** 3·5 dcm **14.** 14 cm., 17·5 cm. **15.** 2849 cu. dcm.

Exercise 7

1. Rs. 9·24 **2.** 2·8 dcm **3.** 10·5 cm **4.** 38·808 c.c., 55·44 sq. cm. **5.** 8 pieces **6.** 8 pieces, 4·851 cu. dcm **7.** Rs. 41·58 **8.** 36 : 25 **9.** 50·4 cm **10.** 1·4 dcm **11.** 8 cm **12.** 2·1 dcm **14.** 4 pieces **15.** 12 dcm.